Graphical Enumeration: A Species-Theoretic Approach

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ABSTRACT

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by Leopold Travis

An operation on species corresponding to the inner plethysm of their associated cycle index series is constructed. This operation, the inner plethysm of species, is generalized to n-sorted species. Polynomial maps on species are studied and used to extend inner plethysm and other operations to virtual species. Finally, inner plethysm and other operations on species are applied to various problems in graph theory.

In particular, regular graphs, and digraphs in which every vertex has outdegree k, are enumerated.

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Chapter 1

Introduction

Since its introduction by Joyal in [6], the concept of *species of structures* has proved useful in many areas of combinatorics. Our aim in this work is two-fold: to study certain new or little-known operations in the theory of species, and to apply these operations to various problems in graphical enumeration.

Chapter 2 is devoted to species. We define the cycle index series (in a slightly different way from Joyal), and relate it to the *analytic functors* he introduces in [7]. We use our techniques to prove the well-known formula for the cycle index series of the composition of two species.

We then investigate the operation of *inner plethysm*. As an operation on symmetric functions it was introduced by Littlewood in [9], and has been used in the study of representation theory (see Thibon [14]). The corresponding operation on species has not been examined. We give two constructions for it, one using combinatorial operations on species, the other using analytic functors.

Next, we examine inner plethysm in the context of n-sorted species. We focus upon the analytic functor approach, using it both to define the operation of *inner plethysm in Y*, and to calculate the corresponding operation on symmetric functions.

Finally, we examine *polynomial maps on species*, using essentially the techniques of [15] to extend our results to virtual species.

In Chapter 3 we study of digraphs. In particular we count unlabeled digraphs in which every vertex has outdegree k, which to our knowledge is an open problem.

In Chapter 4 we study graphs, once again emphasizing a species-theoretic point of view and applying techniques from Chapter 2.

Chapter 2

Species

2.1 The cycle index and symmetric functions

Following [10], we denote by Λ the ring of symmetric functions in the infinite set of indeterminates $x_1, x_2, \ldots$, and by Λ^n the subring of Λ consisting of symmetric functions homogeneous of degree n. More generally, Λ_x will denote the ring of symmetric functions in the indeterminates $x_1, x_2, \ldots$, Λ_y will denote the ring of symmetric functions in $y_1, y_2, \ldots$, etc. We will denote by Λ_{xy} the ring of formal series in $x_1, x_2, \ldots, y_1, y_2, \ldots$ of bounded degree, which are symmetric separately in the x_i and in the y_i. We define Λ_{xyz}, etc., similarly. It is not difficult to show that Λ_{xy} is (isomorphic to) the tensor product $\Lambda_x \otimes \Lambda_y$, $\Lambda_{xyz} \cong \Lambda_x \otimes \Lambda_y \otimes \Lambda_z$, and so forth. We define $\hat{\Lambda}$ to be the ring of symmetric functions in $x_1, x_2, \ldots$ of unbounded degree, and $\hat{\Lambda}_{xy}$, etc., similarly.

Let F be a species, that is, a functor from $\mathbb{B}$ to $\mathbb{B}$, where $\mathbb{B}$ is the category of finite sets and bijections. We denote by $[n]$ the set $\{1, \ldots, n\}$, and write $F[n]$ for $F[\{1, \ldots, n\}]$. The symmetric group $\mathfrak{S}_n$ acts on $F[n]$, since by functoriality any $\sigma \in \mathfrak{S}_n$ induces a permutation $F[\sigma]$ of $F[n]$. For any partition λ of n, we define fix $F[\lambda]$ to be the number of fixed points of $F[\sigma]$, where σ is any permutation of cycle type λ. Following [3], we associate to F its *cycle index series*, or *cycle index*, the symmetric function Z_F:

$$Z_F = \sum_\lambda \text{fix}\, F[\lambda] \frac{p_\lambda}{z_\lambda} \tag{2.1}$$

In this and all similar sums, the index of summation λ is taken to run over the set $\mathcal{P}$ of all partitions.

Our definition of the cycle index differs from that introduced by Joyal in [6] in that we use the power sum symmetric function p_i in place of an indeterminate. Thus our cycle index series can be considered both as a formal series in the power sums p_i, and as a symmetric function in the underlying variables x_i. We will explore the combinatorial significance of this second interpretation in the

next sections. First, we give a slight generalization of (2.1), to weighted species (see [6, Section 6]).

For a commutative ring R containing the rationals, let Sets_R denote the category of R-*weighted sets*. The objects of Sets_R are pairs (A, w), where A is a set and $w : A \to R$ is a function, which we will refer to as a *weight function*. We do not require that A be finite, but $\sum_{a \in A} w(a)$ must exist in R for (A, w) to be an object of Sets_R; we denote this sum by $|A|_w$. A morphism $f : (A, w) \to (B, v)$ is a function $f : A \to B$ which is weight-preserving, i.e., a function f such that $v(f(a)) = w(a)$ for all $a \in A$. By abuse of notation, we will often refer to an object (A, w) of Sets_R simply as A, and write "$A \in \text{Sets}_R$."

An R-*weighted species*, or simply *weighted species* if R is given, is a functor $F : \mathbb{B} \to \text{Sets}_R$. As before, $\mathfrak{S}_n$ acts on $F[n]$, and we define $\text{fix}_w F[\sigma]$ to be the sum of the weights of all elements of $F[n]$ fixed by σ. This sum clearly depends only on the cycle type of σ, so we can define $\text{fix}_w F[\lambda]$ for any $\lambda \vdash n$. We define the *weighted cycle index series* of F to be:

$$Z_F = \sum_\lambda \text{fix}_w F[\lambda] \frac{p_\lambda}{z_\lambda} \tag{2.2}$$

Any species can be considered as a weighted species simply by assigning $w(a) = 1$ for all $a \in F[A]$, for any finite set A. We then have $\text{fix}_w F[\lambda] = \text{fix} F[\lambda]$ for any λ, and (2.1) and (2.2) are identical. We will sometimes refer to a species, considered as a weighted species in this fashion, as an *ordinary* species.

We will use the following notation, much of it standard in the theory of species. By $\mathbf{0}$ (or simply 0 when clear from the context) we denote the *empty species*, $\mathbf{0}[U] = \emptyset$ for all finite sets U. By $\mathbf{1}$ (or 1) we denote the *empty set species*,

$$\mathbf{1}[U] = \begin{cases} \{U\} & \text{if } U = \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

Similarly, we have the species X of *singletons*,

$$X[U] = \begin{cases} \{U\} & \text{if } |U| = 1 \\ \emptyset & \text{otherwise} \end{cases}$$

and the species E_k of sets of cardinality k:

$$E_k[U] = \begin{cases} \{U\} & \text{if } |U| = k \\ \emptyset & \text{otherwise} \end{cases}$$

The *uniform species* E, or *species of sets*, is given by $E[U] = \{U\}$ for all finite sets U. Thus, $E = E_0 + E_1 + E_2 + \cdots = 1 + X + E_2 + \ldots$.

The combinatorial operations of sum $(+)$, product $(\cdot)$, derivation $(')$, and substitution $(\circ)$ on species induce corresponding operations on their cycle index series—i.e., on symmetric functions. The operations corresponding to $+$ and $\cdot$

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are easily described (they are the ordinary sum and product on symmetric functions), and the operation corresponding to $'$ is $\partial\,/\partial p_1$. That is, $Z_{F'} = \partial Z_F/\partial p_1$, where Z_F is given in terms of the p_i by (2.1). The operation corresponding to $\circ$ is well-known as well (see [6, Theorem 3]); in our framework it corresponds to the operation of plethysm of symmetric functions. (We explore substitution and plethysm in more detail in Section 2.1.3.)

The Cartesian product $(\times)$ of species F and G is defined by $(F \times G)[U] = F[U] \times G[U]$ for all finite sets U. The corresponding operation on symmetric functions is the internal, or Kronecker product. Following [3], we will denote it by $\times$ as well. Thus,

$$\sum_\lambda a_\lambda \frac{p_\lambda}{z_\lambda} \times \sum_\mu b_\mu \frac{p_\mu}{z_\mu} = \sum_\lambda a_\lambda b_\lambda \frac{p_\lambda}{z_\lambda}$$

and for species F and G, $Z_{F\times G} = Z_F \times Z_G$.

Closely related to the Cartesian product is the *scalar product* $\langle\,,\rangle$ (see [3, 7]), given by $\langle F, G\rangle = (F \times G)|_{X=1}$. The scalar product on species corresponds to the scalar product on symmetric functions, which is given by:

$$\left\langle \sum_\lambda a_\lambda \frac{p_\lambda}{z_\lambda}, \sum_\mu b_\mu \frac{p_\mu}{z_\mu}\right\rangle = \sum_\lambda \frac{a_\lambda b_\lambda}{z_\lambda} \tag{2.3}$$

These notions generalize to species of several sorts (see [3] for details). For example, the cycle index of a 2-sorted species $F(X, Y)$ is given by:

$$Z_F = \sum_{\lambda,\mu} \text{fix}\, F[\lambda, \mu] \frac{p_\lambda(x)}{z_\lambda} \frac{p_\mu(y)}{z_\mu} \tag{2.4}$$

(Here we follow the notation of [3] for symmetric functions in several sets of variables: given $f \in \Lambda$, $f(y)$ denotes f as a symmetric function of the variables $y_1, y_2, \ldots$, $f(z)$ denotes f in the variables $z_1, z_2, \ldots$, etc. Thus, $p_1(y) = y_1 + y_2 + y_3 + \ldots$, $p_2(z) = z_1{}^2 + z_2{}^2 + \ldots$, and so forth.) For 2-sorted species $F(X, Y)$, $G(X, Y)$, their *Cartesian product in Y* is given by:

$$(F \times_Y G)[U, V] = \sum_{U_1+U_2=U} F[U_1, V] \times G[U_2, V] \tag{2.5}$$

and the cycle index of $F \times_Y G$ is obtained by expressing Z_F and Z_G in terms of the $p_\lambda(y)$, with coefficients in $\hat{\Lambda}_x$:

If

$$Z_F = \sum_\lambda a_\lambda(x) \frac{p_\lambda(y)}{z_\lambda}$$

$$Z_G = \sum_\mu b_\mu(x) \frac{p_\mu(y)}{z_\mu}$$

for some $a_\lambda(x), b_\mu(x)$ in $\hat{\Lambda}_x$, then

$$Z_{F \times_Y G} = \sum_\lambda a_\lambda(x) b_\lambda(x) \frac{p_\lambda(y)}{z_\lambda}$$

The *scalar product in Y* of F and G is then given by $\langle F, G \rangle_Y = (F \times_Y G)|_{Y=1}$, and its cycle index is obtained from $Z_{F \times_Y G}$ by setting each $p_\lambda(y)$ equal to 1. (Inspection of (2.3) shows that this procedure is entirely analogous to that in the 1-sorted case.)

Finally, we recall that both the *exponential generating function* and the *isomorphism-types generating function* of a species F may be obtained from its cycle index series by certain substitutions. In our framework, the exponential generating function $F(x)$ is obtained by setting p_1 equal to x, and p_i equal to 0 for $i > 1$. The isomorphism-types generating function $\widetilde{F}(x)$ is obtained by setting p_i equal to x^i for all i. (For an n-sorted species, one performs these substitutions in each variable. For example, the isomorphism-types generating function $\widetilde{F}(x, y)$ of a 2-sorted species $F(X, Y)$ is obtained from the cycle index (2.4) by setting $p_i(x)$ equal to x^i and $p_i(y)$ equal to y^i for all i.)

2.1.1 Burnside's Lemma

For manipulating and calculating with cycle index series, a generalization of Burnside's Lemma will be vital to us. We state it here. (This result is not new; it is essentially a weighted variation of Lemma 5 in [3].)

Suppose a finite group $G \times H$ acts on a set $S \in \text{Sets}_R$, and $w : S \to R$ is the weight function of S. By this we mean that $G \times H$ acts on the set S, and that for any $(g, h) \in G \times H$, the map $s \mapsto (g, h) \cdot s$ is a morphism in Sets_R. Let $\text{fix}_w (g, h)$ denote the sum of the weights of all elements of S fixed by (g, h). The groups G and H, considered as subgroups of $G \times H$, also act on S, and for $s \in S$, let $\mathcal{O}_G(s)$ denote the orbit of s under the action of G, or simply the "G-orbit" of s.

Lemma 2.1. *The weight function w is constant on each G-orbit, allowing us to define $w(\mathcal{O}_G(s))$ to be $w(s)$. The group H acts on the set of G-orbits, and for any $h \in H$, the sum of the weights of the G-orbits fixed by h is:*

$$\frac{1}{|G|} \sum_{g \in G} \text{fix}_w (g, h) \tag{2.6}$$

Proof. Considering G and H as subgroups of $G \times H$ means identifying G with the set $G \times \{e_H\}$ and H with the set $\{e_G\} \times H$, where e_G and e_H denote the identity elements of G and H, respectively. Thus the actions of G and H upon S are given by $g \cdot s = (g, e_H) \cdot s$, $h \cdot s = (e_G, h) \cdot s$, for $g \in G$, $h \in H$, and $s \in S$. We note that $g \cdot (h \cdot s) = (gh) \cdot s = (hg) \cdot s = h \cdot (g \cdot s)$. (By gh we mean (g, h), since $(g, e_H)(e_G, h) = (g, h)$ in $G \times H$.)

It is immediate that w is constant on each orbit under G, under H, and under $G \times H$, by the definition of morphisms in Sets_R. To show that H acts on the set of G-orbits, define:

$$h \cdot \mathcal{O}_G(s) = \mathcal{O}_G(h \cdot s) \tag{2.7}$$

The action (2.7) is well-defined, for if $\mathcal{O}_G(s_1) = \mathcal{O}_G(s_2)$ then there exists $g \in G$ such that $s_1 = g \cdot s_2$, and,

$$\begin{aligned}
h \cdot \mathcal{O}_G(s_1) &= h \cdot \mathcal{O}_G(g \cdot s_2) \\
&= \mathcal{O}_G(h \cdot (g \cdot s_2)) \\
&= \mathcal{O}_G((hg) \cdot s_2) \\
&= \mathcal{O}_G((gh) \cdot s_2) \\
&= \mathcal{O}_G(g \cdot (h \cdot s_2)) \\
&= \mathcal{O}_G(h \cdot s_2) \\
&= h \cdot \mathcal{O}_G(s_2)
\end{aligned}$$

That (2.7) *is* a group action is clear:

$$\begin{aligned}
h_1 \cdot (h_2 \cdot \mathcal{O}_G(s)) &= \mathcal{O}_G(h_1 \cdot (h_2 \cdot s)) \\
&= \mathcal{O}_G((h_1 h_2) \cdot s) \\
&= (h_1 h_2) \cdot \mathcal{O}_G(s)
\end{aligned}$$

To calculate the sum of the weights of the G-orbits fixed by $h \in H$, we consider the set $\mathcal{S}$ of pairs (g, s), where $g \in G$, $s \in S$, and $g^{-1} \cdot s = h \cdot s$. Define the weight $w(g, s)$ of the pair (g, s) to be $w(s)$. Now $g^{-1} \cdot s = h \cdot s$ if and only if $(g, h) \cdot s = s$, i.e., if and only if (g, h) fixes s. By the definition of fix_w, this means that for a given $g \in G$, the sum of the weights of pairs of the form (g, s) $\mathcal{S}$ is simply $\mathrm{fix}_w(g, h)$. Thus,

$$\sum_{(g,s) \in \mathcal{S}} w(g, s) = \sum_{g \in G} \mathrm{fix}_w(g, h) \tag{2.8}$$

We now calculate this sum of weights in a different way. For a pair $(g, s) \in \mathcal{S}$, consider the number of choices for g, given s. If $\mathcal{O}_G(s)$ is not fixed by h, clearly there are none (because if there exists $g \in G$ such that $g^{-1} \cdot s = h \cdot s$, then $h \cdot s \in \mathcal{O}_G(s)$, and thus $h \cdot \mathcal{O}_G(s) = \mathcal{O}_G(s)$ by (2.7)). If h does fix $\mathcal{O}_G(s)$ then $\mathcal{O}_G(s) = \mathcal{O}_G(h \cdot s)$, so there exists $g_1 \in G$ such that $g_1 \cdot s = h \cdot s$. Now, $g^{-1} \cdot s = h \cdot s \Leftrightarrow g^{-1} \cdot s = g_1 \cdot s \Leftrightarrow s = g g_1 \cdot s \Leftrightarrow g g_1 \in \mathrm{stab}_G(s)$, where $\mathrm{stab}_G(s) = \{g \in G : g \cdot s = s\}$ denotes the G-stabilizer of s. So the number of choices for g is the number of g such that $g g_1 \in \mathrm{stab}_G(s)$, which is simply $|\mathrm{stab}_G(s)|$ (since $g g_1 \in \mathrm{stab}_G(s) \Leftrightarrow g \in \mathrm{stab}_G(s) g_1^{-1}$, and $|\mathrm{stab}_G(s) g_1^{-1}| = |\mathrm{stab}_G(s)|$).

Let $\mathcal{O}_G(s_1), \ldots, \mathcal{O}_G(s_k)$ be the G-orbits fixed by h. Since there are $|\mathrm{stab}_G(s)|$

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choices for g in the pair $(g, s) \in \mathcal{S}$, given s, we have:

$$\sum_{(g,s) \in \mathcal{S}} w(g,s) = \sum_{i=1}^{k} \sum_{s \in \mathcal{O}_G(s_i)} |\text{stab}_G(s)| w(s)$$

$$= \sum_{i=1}^{k} \sum_{s \in \mathcal{O}_G(s_i)} \frac{|G|}{|\mathcal{O}_G(s)|} w(s)$$

$$= \sum_{i=1}^{k} |G| \sum_{s \in \mathcal{O}_G(s_i)} \frac{w(s)}{|\mathcal{O}_G(s)|}$$

The weight function w is constant on orbits, and $|\mathcal{O}_G(s)| = |\mathcal{O}_G(s_i)|$ if s is an element of $\mathcal{O}_G(s_i)$, so,

$$\sum_{i=1}^{k} |G| \sum_{s \in \mathcal{O}_G(s_i)} \frac{w(s)}{|\mathcal{O}_G(s)|} = |G| \sum_{i=1}^{k} w(\mathcal{O}_G(s_i)) \qquad (2.9)$$

by the definition of the weight of an orbit.

Combining (2.8) and (2.9) completes the proof. □

Corollary 2.2 (Burnside's Lemma, weighted form). *Suppose a finite group G acts on a set S, and w is a weight function which is constant on orbits. Let $\text{fix}_w g$ denote the sum of the weights of elements of S fixed by $g \in G$. Then the sum of the weights of the orbits under the action of G is*

$$\frac{1}{|G|} \sum_{g \in G} \text{fix}_w g \qquad (2.10)$$

Proof. Apply Lemma 2.1 to the case where H is a trivial group. □

Corollary 2.2 is of course well-known (see for example (2.3.10) in [5]).

2.1.2 Analytic functors

In [7], Joyal associates to the species F an *analytic functor* $F() : \text{Sets} \to \text{Sets}$, where Sets is the category of sets and functions. His construction is as follows. For a set A, consider the set $F[n] \times A^n$ of all pairs (t, f), where $t \in F[n]$ and f is a function from $[n]$ to A. The symmetric group $\mathfrak{S}_n$ acts on these pairs via:

$$\sigma \cdot (t, f) = (\sigma \cdot t, f \circ \sigma^{-1}) \qquad (2.11)$$

where $\sigma \cdot t$ denotes $F[\sigma](t)$. Let $F[n] \times A^n / \mathfrak{S}_n$ be the set of orbits under this action, and let $\mathcal{O}_{\mathfrak{S}_n}(t, f)$ be the orbit of the pair (t, f). The analytic functor $F()$ is then defined by:

$$F(A) = \sum_{n \geq 0} F[n] \times A^n / \mathfrak{S}_n \qquad (2.12)$$

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for all sets A. (For a function $g : A \to B$, the map $F(g) : F(A) \to F(B)$ is defined by $\mathcal{O}_{\mathfrak{S}_n}(t, f) \mapsto \mathcal{O}_{\mathfrak{S}_n}(t, g \circ f)$. It is easily verified that $F(g)$ is well-defined and preserves composition.)

Remark. The analytic functor $F()$, written with parentheses, must not be confused with the species F, which is itself a functor and is often written $F[\,]$.

We generalize the notion of analytic functor to that of *weighted analytic functor* by modifying the category on which $F()$ is defined. For a weighted species F, we construct the weighted analytic functor $F() : \mathrm{Sets}_R \to \mathrm{Sets}_R$ associated to a species F as follows. For $A \in \mathrm{Sets}_R$, we have the set of pairs $(t, f) \in F[n] \times A^n$, and the orbits $F[n] \times A^n / \mathfrak{S}_n$, as above. Define the weight of f to be:

$$w(f) = \prod_{a \in A} w(a)^{|f^{-1}(a)|} = w(f(1))w(f(2))\dots \tag{2.13}$$

For any $\sigma \in \mathfrak{S}_n$ and $a \in A$, the inverse images of a under f and $f \circ \sigma^{-1}$ have the same size (since $(f \circ \sigma^{-1})^{-1}(a) = \sigma(f^{-1}(a))$, and σ is a permutation), so the weight we have defined is constant on orbits of A^n under $\mathfrak{S}_n$. Define the weight of the pair (t, f) to be $w(t)w(f)$. The functor $F()$ is then defined by equation (2.12), with the weight $w(\mathcal{O}_{\mathfrak{S}_n}(t, f))$ of an orbit defined to be the weight of any of its elements.

Definition 2.3. For a weighted analytic functor F, and $A \in \mathrm{Sets}_R$, we define the *type-series* of $F(A)$ to be:

$$Z_{F(A)} = |F(A)|_w = \sum_{\mathcal{O} \in F(A)} w(\mathcal{O}) \tag{2.14}$$

We now relate the type-series to the cycle index (2.2). Consider the case where $R = \mathbb{Q}[[x_1, x_2, \dots]]$, the ring of formal power series in the variables $x_1, x_2, \dots$, with coefficients in $\mathbb{Q}$. We consider the set $\mathbb{P} = \{1, 2, \dots\}$ of positive integers to be an element of Sets_R by assigning it the following weight function: for $i \in \mathbb{P}$, let $w(i) = x_i$. (For any $S \subseteq \mathbb{P}$, we define a weight function on S by restriction.) By (2.13), the weight of a function $f : [n] \to \mathbb{P}$ is the following monomial in the x_i:

$$w(f) = x_1^{|f^{-1}(1)|} x_2^{|f^{-1}(2)|} \dots \tag{2.15}$$

We will often refer to f as a "coloring" of the elements of $[n]$, and think of (t, f) as the F-structure t together with a coloring of the elements of its underlying set $[n]$. An orbit under the action $\mathfrak{S}_n$—i.e., an element of $F(\mathbb{P})$—thus corresponds to an "unlabeled, colored F-structure," or an "isomorphism class of colorings of F-structures."

In order to count the orbits under $\mathfrak{S}_n$ by weight, we will use Burnside's Lemma. To apply it, we must calculate the sum of the weights of pairs (t, f) fixed by a permutation $\sigma \in \mathfrak{S}_n$.

Suppose σ has cycle type $\lambda = (1^{m_1} 2^{m_2} \dots)$. If σ fixes a pair (t, f) then by (2.11), σ must fix t (that is, $F[\sigma](t) = t$), and f must be constant on each cycle of σ. If a cycle c of length l receives color i (that is, if $f(j) = i$ for all j in the cycle c), this results in a factor of $x_i{}^l$ in the weight of (t, f). The colors can be assigned arbitrarily, and there are m_l cycles of length l, the sum of the weights of the colorings of a given F-structure t fixed by σ is:

$$(x_1 + x_2 + \dots)^{m_1} (x_1{}^2 + x_2{}^2 + \dots)^{m_2} (x_1{}^3 + x_2{}^3 + \dots)^{m_3} \dots \qquad (2.16)$$

since each term in the expansion of this product corresponds to the weight of a specific coloring.

The product (2.16) is immediately expressible in terms of power sum symmetric functions as $p_1{}^{m_1} p_2{}^{m_2} \cdots = p_\lambda$. Thus, we find that the sum of the weights of pairs (t, f) fixed by σ of cycle type λ is:

$$\mathrm{fix}_w \, \sigma = \mathrm{fix}_w \, F[\sigma] p_\lambda$$

Applying Corollary 2.2, the sum of the weights of orbits under the action (2.11) is:

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{fix}_w \, F[\sigma] p_\lambda \qquad (2.17)$$

where λ is the cycle type of σ. Using the fact that there are $n!/z_\lambda$ permutations of cycle type λ in $\mathfrak{S}_n$, we can rewrite (2.17) as:

$$\sum_{\lambda \vdash n} \mathrm{fix}_w \, F[\lambda] \frac{p_\lambda}{z_\lambda} \qquad (2.18)$$

In light of (2.18), the combinatorial interpretation of (2.1) is now clear: the cycle index of a species F, considered as a formal series in the x_i, counts by weight all isomorphism classes of colorings of F-structures. (The sum (2.18) counts only those on underlying sets of cardinality n.) The connection with the type-series is:

$$Z_{F(\mathbb{P})} = Z_F \qquad (2.19)$$

where the Z_F on the right-hand side is the weighted cycle index (2.2), considered as a formal series in the x_i.

Remark. The connection between (2.1) and (2.17), with the p_i replaced by formal variables, is of course well-known (see for example [6, Proposition 13]). In fact, the equality (2.19) could also have been proven by Pólya theory. We recall that given a subgroup G of $\mathfrak{S}_n$, its cycle index polynomial $Z(G)$ is given by (using the p_i in place of formal variables):

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} p_\lambda$$

where λ is the cycle type of g. By Proposition 13 of [6], the cycle index Z_F of a species F can be expressed as:

$$Z_F = \sum_t Z(\mathrm{Aut}(t)) \tag{2.20}$$

where the sum is over a set of representatives of all isomorphism classes of F-structures, and $\mathrm{Aut}(t)$ denotes the automorphism group of an F-structure t. By Pólya's theorem (see, for example, (2.4.16) in [5] for a statement), replacing each p_i by $x_1{}^i + x_2{}^i + \cdots + x_n{}^i$ in (2.20) gives a formal series which counts by weight the inequivalent colorings of F-structures, colored in n colors. Since this holds for any positive n, we obtain the equality (2.19).

2.1.3 Composition of species

As an application of our interpretation of the cycle index, we provide a proof of the well-known expression for the cycle index of the composition of two species. We first recall the operation $\circ$ of *plethysm* (see [10, Section I.8]) on symmetric functions. Plethysm satisfies $(fg) \circ h = (f \circ h)(g \circ h)$, $(f + g) \circ h = f \circ h + g \circ h$ for all $f, g, h \in \Lambda$. The plethysm $p_n \circ g$ is obtained from the expression for g (in terms of the x_i) by replacing each x_i with $x_i{}^n$. For example, it follows immediately that $p_n \circ p_m = p_{nm}$ for all $n, m \in \mathbb{N}$. (We will deal with plethysm in more detail, in the context of λ-rings, in Section 2.2.)

Lemma 2.4. *Let F and G be ordinary species, considered as weighted species, with associated weighted analytic functors $F()$, $G()$. Then:*

$$Z_{F(G(\mathbb{P}))} = Z_{F(\mathbb{P})} \circ Z_{G(\mathbb{P})} \tag{2.21}$$

Proof. By definition, an element of $F(G(\mathbb{P}))$ is the orbit under $\mathfrak{S}_n$, for some n, of a pair (t, f), where $t \in F[n]$ and $f : [n] \to G(\mathbb{P})$. The weight of the pair (t, f) is, by (2.13):

$$w(t, f) = \prod_{\mathcal{O} \in G(\mathbb{P})} w(\mathcal{O})^{|f^{-1}(\mathcal{O})|}$$

Suppose $\sigma \in \mathfrak{S}_n$ fixes (t, f), where σ has cycle type $\lambda = (1^{m_1} 2^{m_2} \dots)$. Then σ must fix t, and since $f \circ \sigma^{-1} = f$, f must be constant on the cycles of σ. Suppose the value of f on a cycle c of length l in σ is $\mathcal{O} \in G(\mathbb{P})$. The contribution to the weight of (t, f) will be a factor of $w(\mathcal{O})^l$. Since $w(\mathcal{O})$ is a monomial (by (2.15)), $w(\mathcal{O})^l$ can be obtained from $w(\mathcal{O})$ by replacing each x_j in $w(\mathcal{O})$ with $x_j{}^l$. Thus, given t, the sum of the weights of pairs (t, f) fixed by σ is:

$$(p_1 \circ Z_{G(\mathbb{P})})^{m_1} (p_2 \circ Z_{G(\mathbb{P})})^{m_2} \cdots = p_\lambda \circ Z_{G(\mathbb{P})} \tag{2.22}$$

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since each term in the expansion of this product corresponds to the weight of such a pair. Therefore, $\text{fix}_w\, \sigma$ is simply $\text{fix}\, F[\sigma]$ times the product (2.22), and by (2.10), the sum of the weights of the orbits under $\mathfrak{S}_n$ is:

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} p_\lambda \circ Z_{G(\mathbb{P})}$$

where λ is the cycle type of σ.

Summing over all n and applying the fact that there are $n!/z_\lambda$ permutations of cycle type λ in $\mathfrak{S}_n$, we find that:

$$Z_{F(G(\mathbb{P}))} = \sum_\lambda \frac{\text{fix}\, F[\lambda]}{z_\lambda} p_\lambda \circ Z_{G(\mathbb{P})}$$

$$= Z_{F(\mathbb{P})} \circ Z_{G(\mathbb{P})}$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.5. *Let F and G be weighted species with associated weighted analytic functors $F()$, $G()$, and let $(F \circ G)()$ be the weighted analytic functor associated to the species $F \circ G$. There is a natural isomorphism between the functors $(F \circ G)()$ and $F(G())$.*

Proof. We must construct a weight-preserving bijection between $(F \circ G)(A)$ and $F(G(A))$ for any finite set A. Consider first $(F \circ G)(A)$.

An element of $(F \circ G)(A)$ consists of the orbit of a pair (t, f), where $t \in (F \circ G)[m]$ and $f : [m] \to A$, for some m. By the definition of the composition of two species, t consists of:

(1) A partition $U_1 + \cdots + U_n = [m]$, for some n

(2) G-structures $\alpha_1, \ldots, \alpha_n$, with $\alpha_i \in G[U_i]$

(3) An F-structure $\alpha \in F[\{\alpha_1, \ldots, \alpha_n\}]$

We can thus map the orbit of (t, f) to the orbit of an element (t', f') as follows. Let $\gamma : \{\alpha_1, \ldots, \alpha_n\} \to [n]$ be any bijection, and let $\overline{\sigma} \in \mathfrak{S}_n$ be the permutation defined by $\overline{\sigma}(i) = k \Leftrightarrow \gamma(\alpha_k) = i$. For $1 \leq i \leq n$, let $m_i = |U_i|$, and let $\gamma_i : U_i \to [m_i]$ also be bijections. Define $t' = F[\gamma](\alpha)$, and for $1 \leq i \leq n$, let $f' : [n] \to G(A)$ be defined by $f'(i) = \mathcal{O}_{\mathfrak{S}_{m_k}}(G[\gamma_k](\alpha_k), f \circ \gamma_k^{-1})$ for $1 \leq i \leq n$, where $k = \overline{\sigma}(i)$.

Let Γ be the map we have constructed. We first show that Γ is well-defined.

The orbit of (t', f') does not depend on the choice of the bijections γ and $\gamma_1, \ldots, \gamma_n$, for any other bijections will be of the form $\sigma \circ \gamma, \sigma_1 \circ \gamma_1, \ldots, \sigma_n \circ \gamma_n$ for some permutations $\sigma \in \mathfrak{S}_n$, $\sigma_i \in \mathfrak{S}_{m_i}$, $1 \leq i \leq n$. Replacing γ by $\sigma \circ \gamma$ means replacing $\overline{\sigma}$ with $\overline{\sigma} \circ \sigma^{-1}$, since $(\overline{\sigma} \circ \sigma^{-1})(i) = k \Leftrightarrow \gamma(\alpha_k) = \sigma^{-1}(i) \Leftrightarrow (\sigma \circ \gamma)(\alpha_k) = i$. Thus, using these new bijections maps the orbit of (t, f) to the orbit of (t_1', f_1'), where $t_1' = F[\sigma \circ \gamma](\alpha)$ and $f_1'(i) = \mathcal{O}_{\mathfrak{S}_{m_i}}(G[\sigma_k \circ \gamma_k](\alpha_k), f \circ$

$\gamma_k{}^{-1} \circ \sigma_k{}^{-1}$), where $k = (\overline{\sigma} \circ \sigma^{-1})(i)$. But by the definition (2.11) of the actions of the various symmetric groups, it is immediate that $\mathcal{O}_{\mathfrak{S}_n}(t', f') = \mathcal{O}_{\mathfrak{S}_n}(t_1', f_1')$.

The orbit of (t', f') also does not depend on the choice of the representative (t, f), for any other representative will be of the form $(\sigma \cdot t, f \circ \sigma^{-1})$ for some $\sigma \in \mathfrak{S}_m$. The element $\sigma \cdot t$ will consist of:

(1) The partition $\sigma(U_1) + \cdots + \sigma(U_n) = [m]$

(2) The G-structures $G[\sigma|_{U_1}](\alpha_1), \ldots, G[\sigma|_{U_n}](\alpha_n)$

(3) The F-structure $F[\delta](\alpha)$, where δ is the map $\alpha_i \mapsto G[\sigma|_{U_i}](\alpha_i)$ for $1 \le i \le n$

Let γ' be the bijection $\gamma \circ \delta^{-1}$, and γ_i' be $\gamma_i \circ (\sigma|_{U_i})^{-1}$ for $1 \le i \le n$. Using these bijections, one verifies immediately that Γ maps the orbit of $(\sigma \cdot t, f \circ \sigma^{-1})$ to that of (t', f').

To show that Γ is injective, suppose the orbit of (u, g) also maps to the orbit of (t', f'). Then u must consist of:

(1) A partition $U_1' + \cdots + U_n' = [m]$

(2) G-structures $\alpha_1', \ldots, \alpha_n'$, with $\alpha_i' \in G[U_i']$.

(3) An F-structure $\alpha' \in F[\{\alpha_1', \ldots, \alpha_n'\}]$

Let $\gamma', \gamma_1', \ldots, \gamma_n'$ be the bijections used in the definition of Γ to calculate the image of (u, g), and $\overline{\sigma}'$ the permutation corresponding to γ'—i.e., $\overline{\sigma}'(i) = k \Leftrightarrow \gamma'(\alpha_k') = i$. Without loss of generality—since we have shown that the bijections used in the definition of Γ can be chosen arbitrarily—we may assume that $\overline{\sigma}'$ is the identity. Thus, $\gamma'(\alpha_i') = i$ for $1 \le i \le n$. Let $m_i' = |U_i'|$.

Using the bijections $\gamma', \gamma_1', \ldots, \gamma_n'$ in the definition of Γ gives a representative (u', g') of the orbit which (u, g) maps to, and since this is also the orbit to which (t, f) maps, this representative must be of the form $(\sigma \cdot t', f' \circ \sigma^{-1})$ for some $\sigma \in \mathfrak{S}_n$. Therefore, $F[\gamma'](\alpha') = F[\sigma \circ \gamma](\alpha)$. Again because the choice of bijections is arbitrary, we may replace γ by $\sigma \circ \gamma$ and assume $F[\gamma'](\alpha') = F[\gamma](\alpha)$. Recalling that $\gamma'(\alpha_i') = i$ and that $\gamma(\alpha_k) = i \Leftrightarrow \overline{\sigma}(i) = k$, we see that:

$$\gamma^{-1} \circ \gamma' : \alpha_i' \mapsto \alpha_{\overline{\sigma}(i)}$$
$$F[\gamma^{-1} \circ \gamma'](\alpha') = \alpha$$

Moreover, because $(u', g') = (t', f')$, we have that $g' = f'$, and thus:

$$\mathcal{O}_{\mathfrak{S}_{m_i'}}(G[\gamma_i'](\alpha_i'), g \circ (\gamma_i')^{-1}) = \mathcal{O}_{\mathfrak{S}_{m_k}}(G[\gamma_k](\alpha_k), f \circ \gamma_k{}^{-1})$$
$$\text{where } k = \overline{\sigma}(i)$$

The γ_k may be chosen arbitrarily up to the action of a permutation, so we may in fact assume that $G[\gamma_i'](\alpha_i') = G[\gamma_k](\alpha_k)$ and $g \circ (\gamma_i')^{-1} = f \circ \gamma_k{}^{-1}$ for $1 \le i \le n$.

Consider the maps $\gamma_k{}^{-1} \circ \gamma_i'$ from U_i' to U_k. Glueing these maps together yields a permutation τ of $[m]$, and it is straightforward to verify that $\tau \cdot (u, g) = (t, f)$.

To show that Γ is surjective we construct, given (t', f'), an element (t, f) which maps to it. For $1 \leq i \leq n$, choose α_i' in the orbit $f'(i)$. Thus, α_i' consists of a pair (t_i', f_i'), with $t_i' \in G[m_i]$, $f_i' : [m_i] \to A$, for some m_i. Let $U_1 + \cdots + U_n$ be any partition of $m = m_1 + \ldots m_n$ such that $|U_i| = m_i$, and choose $\gamma_i : U_i \to [m_i]$ to be any bijections. Let $\alpha_i = G[\gamma_i^{-1}](\alpha_i')$, and define $f_i = f_i' \circ \gamma_i$. Let $\gamma : \{\alpha_1, \ldots, \alpha_n\}$ be any bijection, and let $t = F[\gamma^{-1}](t')$. Define $f : [m] \to A$ by glueing together the f_i. Then by construction, the orbit of (t, f) maps to the orbit of (t', f') under Γ.

The verification that Γ is weight-preserving, and of naturality, is straightforward. $\quad\square$

Theorem 2.6. *Let F and G be ordinary species. Then*

$$Z_{F \circ G} = Z_F \circ Z_G \tag{2.23}$$

Proof. This follows immediately from (2.19) and Lemmas 2.4 and 2.5, since:

$$\begin{aligned}
Z_{F \circ G} &= Z_{(F \circ G)(\mathbb{P})} \\
&= Z_{F(G(\mathbb{P}))} \\
&= Z_{F(\mathbb{P})} \circ Z_{G(\mathbb{P})} \\
&= Z_F \circ Z_G
\end{aligned}$$

$\square$

2.2 The inner plethysm of symmetric functions

We first review some facts about representation theory. By a *representation* we shall always mean a complex, finite-dimensional representation, i.e., a homomorphism from a finite group G to $\mathrm{GL}(n, \mathbb{C})$ for some $n \geq 0$. We take our notation for λ-rings from [8]. In particular, given a λ-ring R, $\psi^n : R \to R$ will denote the nth Adams operation of R.

Given a finite group G, $\mathrm{R}(G)$ will denote the *representation ring* of G, that is, the integer span of all isomorphism classes of representations of G, with addition given by direct sum, and multiplication by tensor product. For a representation ρ of G, $[\rho] \in \mathrm{R}(G)$ will denote its isomorphism class. The representation ring is a λ-ring via the operations:

$$\lambda^n([\rho]) = [\wedge^n(\rho)]$$

where $\wedge^n$ denotes the nth exterior power.

We denote by $\mathrm{CF}(G)$ the ring of all central functions of G, that is, the ring of all functions from G to $\mathbb{C}$ which are constant on conjugacy classes. (Addition

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and multiplication are defined via pointwise addition and multiplication in $\mathbb{C}$.)
The λ-ring structure of $\mathrm{CF}(G)$ is given by its Adams operations,

$$(\psi^k(c))(g) = c(g^k) \tag{2.24}$$

for all $c \in \mathrm{CF}(G)$ and $g \in G$.

There is a map $\chi : \mathrm{R}(G) \to \mathrm{CF}(G)$ which sends $[\rho] \in \mathrm{R}(G)$ to the character of ρ, $\chi_\rho = \mathrm{tr} \circ \rho$. (Here tr denotes trace.) As shown in [8], χ sends $\mathrm{R}(G)$ isomorphically onto its image, which we will denote here by $\mathrm{CR}(G)$, the *character ring* of G. The irreducible characters (that is, images of irreducible representations) form a $\mathbb{Z}$-basis for $\mathrm{CR}(G)$, and a $\mathbb{C}$-basis for $\mathrm{CF}(G)$.

If G is the symmetric group $\mathfrak{S}_n$, we can say more. As shown in [10], $\mathrm{CR}(\mathfrak{S}_n)$ is isomorphic to Λ^n, the ring of symmetric functions homogeneous of degree n (where the multiplication in Λ^n is taken to be the Kronecker product). The isomorphism is given by the characteristic map ch,

$$\mathrm{ch}(\tau) = \sum_{\lambda \vdash n} \tau_\lambda \frac{p_\lambda}{z_\lambda} \tag{2.25}$$

where τ_λ denotes the value of $\tau \in \mathrm{CR}(\mathfrak{S}_n)$ on permutations of cycle type λ.

Given a λ-ring R, the ring Λ of symmetric function acts upon R in a natural way. The construction is given by Knutson (see [8, page 46]) as follows. For $f \in \Lambda$, we can express f as a polynomial F in the elementary symmetric functions e_i: $f = F(e_1, e_2, \dots)$. The symmetric function f then becomes a natural operation on R via $r \mapsto F(\lambda(r), \lambda^2(r), \lambda^3(r), \dots)$ for all $r \in R$.

Definition 2.7. For f and r as above, we will denote $F(\lambda(r), \lambda^2(r), \lambda^3(r), \dots)$ by $f[r]$.

Remark. We can define an action of $\hat{\Lambda}$ upon a λ-ring similarly. However, an element $f \in \hat{\Lambda}$ is in general a formal series $F(e_1, e_2, \dots)$, and thus $f[r]$ may not be defined.

Lemma 2.8. *Let a and b be elements of a λ-ring R, and $n, m \geq 1$. Then,*

(1) $p_n[r] = \psi^n(r)$

(2) $p_n[a + b] = p_n[a] + p_n[b]$

(3) $p_n[ab] = p_n[a]p_n[b]$

(4) $p_n[p_m[a]] = p_{nm}[a]$

Proof. Item (1) follows immediately from the definition of the Adams operations (see [8, page 47]). Items (2) – (4) then follow from the properties of ψ^n (see [8, page 48]). $\qquad\square$

Example. The most natural application of Definition 2.7 is the case in which R is Λ itself (since Λ is the free λ-ring on one generator). In this case, for $f, g \in \Lambda$, $f[g]$ is simply the plethysm $f \circ g$.

We describe the inner plethysm of symmetric functions via Definition 2.7 as well. Set $R = \mathrm{CR}(\mathfrak{S}_n)$ and denote by $\boxtimes$ the action of Λ on Λ^n induced by identifying $\mathrm{CR}(\mathfrak{S}_n)$ and Λ^n via the isomorphism (2.25). For $f \in \Lambda$, $g \in \Lambda^n$, $f \boxtimes g$ is the *inner plethysm* of f and g. To extend $\boxtimes$ to an action $\Lambda \times \Lambda \to \Lambda$, we define $f \boxtimes (g_1 + g_2) = f \boxtimes g_1 + f \boxtimes g_2$ for $g_1 \in \Lambda^n, g_2 \in \Lambda^m$ with $n \neq m$.

Inner plethysm corresponds to the composition of a representation of $\mathfrak{S}_n$ with a representation of a general linear group (indeed, this is how it was originally defined by Littlewood [9]).

Let $\phi : \mathrm{GL}(m, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ be a representation. As in [13], if ϕ is a polynomial representation, we associate to ϕ its character f_ϕ, that is, the symmetric function $f(x_1, \ldots, x_n)$ such that:

$$f(\theta_1, \ldots, \theta_n) = \mathrm{tr}\,(\phi(A))$$

for any $A \in GL(m, \mathbb{C})$, where $\theta_1, \ldots, \theta_n$ are the eigenvalues of A.

We note that f_ϕ is a symmetric function in a finite number of variables. However, by abuse of notation, we will also denote by f_ϕ any element $f \in \Lambda$ such that $f(x_1, \ldots, x_n, 0, 0, \ldots) = f_\phi(x_1, \ldots, x_n)$.

Theorem 2.9. *Let* $\rho : \mathfrak{S}_n \to \mathrm{GL}(m, \mathbb{C})$ *and* $\phi : \mathrm{GL}(m, \mathbb{C}) \to \mathrm{GL}(q, \mathbb{C})$ *be representations. Then,*

$$\chi_{\phi \circ \rho} = f_\phi[\chi_\rho]$$

in the λ-*ring* $\mathrm{CR}(\mathfrak{S}_n)$, *the action of* Λ *on* $\mathrm{CR}(\mathfrak{S}_n)$ *being given by Definition 2.7.*

Proof. Fix $\sigma \in \mathfrak{S}_n$, and let $M = \rho(\sigma)$. Suppose M has eigenvalues $\theta_1, \ldots, \theta_m$. Then $\chi_\rho(\sigma) = \theta_1 + \cdots + \theta_m$, by definition, and since the trace of the kth power of a matrix is the sum of the kth powers of the eigenvalues, $\chi_\rho(\sigma^k) = \theta_1{}^k + \cdots + \theta_m{}^k$. By (2.24) and Definition 2.7, this means:

$$(p_k[\chi_\rho])(\sigma) = \theta_1{}^k + \cdots + \theta_m{}^k$$
$$= p_k(\theta_1, \ldots, \theta_k)$$

It follows immediately from Definition 2.7 that $(p_\lambda[\chi_\rho])(\sigma) = p_\lambda(\theta_1, \ldots, \theta_k)$ for any partition λ.

Now express f_ϕ in terms of the p_λ: $f_\phi = \sum_\lambda a_\lambda p_\lambda$ for some coefficients $a_\lambda \in \mathbb{Q}$. Then,

$$\mathrm{tr}\,\phi(M) = \sum_\lambda a_\lambda p_\lambda(\theta_1, \ldots, \theta_k)$$
$$= \sum_\lambda a_\lambda (p_\lambda[\chi_\rho])(\sigma) \qquad (2.26)$$
$$= f_\phi[\chi_\rho](\sigma)$$

This proves the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

It follows immediately from our definition of inner plethysm that with ρ and ϕ as in (2.26),

$$\mathrm{ch}(\chi_{\phi \circ \rho}) = f_\phi \boxtimes \mathrm{ch}(\chi_\rho)$$

Thus, the inner plethysm of symmetric functions corresponds to the composition of a representation of a symmetric group with a representation of a general linear group.

2.3 The inner plethysm of species

We will now define an operation of inner plethysm on species, such that the cycle index of the inner plethysm of two species is the inner plethysm of their cycle indices. In effect, this will give a combinatorial interpretation to the operation of inner plethysm on symmetric functions. We give two constructions of the operation, one using combinatorial operations on species, the other using the analytic functors introduced in Section 2.1.2.

2.3.1 Combinatorial construction

In order to define the inner plethysm of species, we first study a certain map Φ from 1-sorted species to 2-sorted species. Let $\mathcal{H}(X, Y)$ be the 2-sorted species defined by letting $\mathcal{H}[U, V]$ be the set of all functions from U to V. For a 1-sorted species F, define $\Phi(F)(X, Y)$ by:

$$\Phi(F)[U, V] = \mathcal{H}[U, F[V]] \tag{2.27}$$

Remark. We note that $\Phi(0) = E(Y)$. This is because $\Phi(0)[U, V] = \emptyset$ if $U \neq \emptyset$, but if $U = \emptyset$ and V is any set then $\Phi(0)[U, V]$ contains a single element: the empty function from $\emptyset$ to $\emptyset$.

Lemma 2.10.

$$\mathrm{fix}\, \Phi(F)[\beta, \sigma] = \prod_{k \geq 1} (\mathrm{fix}\, F[\sigma^k])^{\beta_k}$$

where β_k denotes the number of cycles of length k of the permutation β.

Proof. Suppose a function f is fixed by $\Phi(F)[\beta, \sigma]$. Then f satisfies $F[\sigma] \circ f \circ \beta^{-1} = f$. The image of every element of a cycle c of β of length l under f is thus determined by the image of some element $i \in c$; moreover, $F[\sigma^l]$ must fix $f(i)$. So the total number of possibilities for f is:

$$\prod_c \mathrm{fix}\, F[\sigma^{l(c)}]$$

where the product is over all cycles c of β, and $l(c)$ is the length of c. This is equivalent to the statement of the lemma. $\qquad\square$

The cycle index series of $\Phi(F)$ is thus:

$$
\begin{aligned}
Z_{\Phi(F)} &= \sum_{\lambda,\mu} \left(\prod_{k \geq 1} \operatorname{fix} F[\sigma^k]^{\beta_k} \right) \frac{p_\lambda(x)}{z_\lambda} \frac{p_\mu(y)}{z_\mu} \\
&= \sum_\mu \exp\left(\sum_{i \geq 1} \frac{\operatorname{fix} F[\sigma^i] p_i(x)}{i} \right) \frac{p_\mu(y)}{z_\mu}
\end{aligned}
\tag{2.28}
$$

where β_k is the number of k-cycles of a permutation of cycle type λ, and σ is a permutation of cycle type μ.

Remark. Lemma 2.10 could also have been proven by noting that $\mathcal{H}(X,Y) = E(E(X) \cdot Y)$ and using this to calculate $Z_\mathcal{H}$, then applying the techniques in [2] (which easily extend to the multi-variable case) to calculate Z_Φ.

We now define the inner plethysm of 1-sorted species, which we will denote $\boxtimes$.

Definition 2.11. For 1-sorted species F, G,

$$
(F \boxtimes G)(Y) = \langle F(X), \Phi(G)(X,Y) \rangle_X
$$

We note that $F \boxtimes G$ is defined if and only if F is *strictly finite* (in the terminology of [15]), i.e., if and only if there exists an integer N such that $F[n] = \emptyset$ whenever $n > N$.

Theorem 2.12.

$$
Z_{F \boxtimes G} = Z_F \boxtimes Z_G \tag{2.29}
$$

Proof. By linearity, it is sufficient to consider the case where G is homogeneous of degree m, for some m. Then G gives a permutation representation of $\mathfrak{S}_m$ with character χ,

$$
\chi(\sigma) = \operatorname{fix} G[\sigma]
$$

for $\sigma \in \mathfrak{S}_m$.

Suppose λ is a partition, $\lambda = (1^{\beta_1} 2^{\beta_2} \ldots)$. Then, in the λ-ring $\mathrm{CR}(\mathfrak{S}_m)$, with the action of Λ on $\mathrm{CR}(\mathfrak{S}_m)$ given by Definition 2.7,

$$
(p_\lambda[\chi])(\sigma) = \chi(\sigma)^{\beta_1} \chi(\sigma^2)^{\beta_2} \chi(\sigma^3)^{\beta_3} \ldots
$$

for $\sigma \in \mathfrak{S}_m$. Therefore, by (2.28),

$$
Z_{\Phi(G)} = \sum_\lambda \sum_{\mu \vdash m} (p_\lambda[\chi])(\sigma) \frac{p_\lambda(x)}{z_\lambda} \frac{p_\mu(y)}{z_\mu}
$$

where σ has cycle type μ.

Let

$$Z_F = \sum_\lambda f_\lambda \frac{p_\lambda(x)}{z_\lambda}$$

Then

$$Z_F \times_X Z_{\Phi(G)} = \sum_\lambda \left(\sum_{\mu \vdash m} (p_\lambda[\chi])(\sigma) \frac{p_\mu(y)}{z_\mu} \right) f_\lambda \frac{p_\lambda(x)}{z_\lambda} \qquad (2.30)$$

where σ has cycle type μ.

Setting $p_\lambda(x) = 1$ in (2.30) for all λ shows that the species $F \boxtimes G$ gives a representation of $\mathfrak{S}_m$ whose character τ is:

$$\tau(\sigma) = \sum_\lambda f_\lambda \frac{(p_\lambda[\chi])(\sigma)}{z_\lambda}$$
$$= (Z_F[\chi])(\sigma) \qquad (2.31)$$

for $\sigma \in \mathfrak{S}_m$. But χ is the character of the representation of $\mathfrak{S}_m$ corresponding to G, i.e. $\mathrm{ch}(\chi) = Z_G$. So by the definition of inner plethysm, (2.31) means that $\mathrm{ch}(\tau) = Z_F \boxtimes Z_G$. Since τ is the character of the representation of $\mathfrak{S}_m$ corresponding to $F \boxtimes G$, $\mathrm{ch}(\tau) = Z_{F \boxtimes G}$, and the theorem is proved. $\square$

2.3.2 Construction by analytic functors

Given species F and G, we have associated functors $F()$ and $G()$ such that the functorial composition of the latter corresponds to the composition, in the species-theoretic sense, of the former. Since F and G are themselves functors, we can consider other compositions; $F(G[])$, for example. We note that $F(G[])$ is a species if and only if F is strictly finite; in this case, $F(U)$ is finite for any finite set U, and thus $F(G[A])$ is finite for any finite set A.

Lemma 2.13. *There is a natural isomorphism between the functors $F(G[])$ and $F \boxtimes G$, where $\boxtimes$ is the operation on species constructed in Section 2.3.1.*

Proof. Let A be a finite set. An element of $F(G[A])$ consists, for some n, of the orbit under $\mathfrak{S}_n$ of a pair (t, f), where $t \in F[n]$, and $f : [n] \to G[A]$. The action of $\mathfrak{S}_n$ is $\sigma \cdot (t, f) = (\sigma \cdot t, f \circ \sigma^{-1})$.

By Definition 2.11, an element of $(F \boxtimes G)[A]$ consists, for some n, of the orbit under $\mathfrak{S}_n$ of a pair (t, f), where $t \in F[n]$, and $f \in \Phi(G)[n, A]$. By (2.27), this means f is a function from $[n]$ to A. The action of $\mathfrak{S}_n$ is $\sigma \cdot (t, f) = (\sigma \cdot t, f \circ \sigma^{-1})$.

The lemma follows immediately. $\square$

Thus, we find that the operation of inner plethysm arises extremely naturally in the context of analytic functors. Indeed, focusing on analytic functors as the objects of study, we could define the inner plethysm of F and G to be $F(G[])$. To show that this point of view is useful, we must show that this definition would provide a means of calculating the cycle index of $F \boxtimes G$.

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Theorem 2.14.

$$Z_{F \boxtimes G} = Z_F \boxtimes Z_G \tag{2.32}$$

Proof. Let H be the species $F(G[\,])$. By (2.19), to calculate the cycle index of $F(G[\,])$, it will be sufficient to calculate the sum of the weights of the elements of $H(\mathbb{P})$.

An element of $H(\mathbb{P})$ consists of the orbit under $\mathfrak{S}_n$ of a pair (t, f), where $t \in H[n]$, $f : [n] \to \mathbb{P}$, for some n. This means $t \in F(G[n])$, and so t is the orbit under $\mathfrak{S}_m$ of (t_1, f_1), where $t_1 \in F[m]$, $f_1 : [m] \to G[n]$, for some m.

We therefore have an action of $\mathfrak{S}_m \times \mathfrak{S}_n$ on the set $\mathcal{S}$ of triples (t_1, f_1, f), given by:

$$(\tau, \sigma) \cdot (t_1, f_1, f) = (\tau \cdot t_1, G[\sigma] \circ f_1 \circ \tau^{-1}, f \circ \sigma^{-1}) \tag{2.33}$$

for $(\tau, \sigma) \in \mathfrak{S}_m \times \mathfrak{S}_n$. The symmetric groups $\mathfrak{S}_m$ and $\mathfrak{S}_n$ act separately as subgroups of $\mathfrak{S}_m \times \mathfrak{S}_n$, and an orbit under $\mathfrak{S}_m$ corresponds to a pair (t, f). The symmetric group $\mathfrak{S}_n$ acts on these orbits, and an orbit of $\mathfrak{S}_m$-orbits under $\mathfrak{S}_n$ corresponds to an element of $H(\mathbb{P})$.

To calculate the sum of the weights of orbits of pairs (t, f) under $\mathfrak{S}_n$ by Burnside's Lemma, we must calculate $\mathrm{fix}_w \, \sigma$ for $\sigma \in \mathfrak{S}_n$. By the previous paragraph, this amounts to calculating the sum of the weights of $\mathfrak{S}_m$-orbits of $\mathcal{S}$ fixed by σ. By Lemma 2.1, we have:

$$\mathrm{fix}_w \, \sigma = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \mathrm{fix}_w \, (\tau, \sigma) \tag{2.34}$$

where $\mathrm{fix}_w \, (\tau, \sigma)$ is the sum of the weights of elements of $\mathcal{S}$ fixed by (τ, σ).

By (2.33), if (t_1, f_1, f) is fixed by (τ, σ), we must have:

(1) $F[\tau](t_1) = t_1$

(2) $G[\sigma] \circ f_1 \circ \tau^{-1} = f_1$

(3) $f \circ \sigma^{-1} = f$

So the number of choices for t_1 in such a triple is simply $\mathrm{fix} \, F[\tau]$, by item (1) above. By item (3), the weight of the triple is p_λ, where the partition λ is the cycle type of σ—by the same argument used to derive (2.16).

We examine item (2). If f_1 satisfies $G[\sigma] \circ f_1 \circ \tau^{-1} = f_1$, and c is a cycle of τ of length l, with $i \in c$, it is immediate that the image of any element of c under f_1 is determined by the image $f_1(i)$. Moreover, $G[\sigma^l]$ must fix $f_1(i)$. So we see that the total number of choices for f_1 is:

$$\prod_c \mathrm{fix} \, G[\sigma^{l(c)}] \tag{2.35}$$

where the product is over all cycles c of τ, and $l(c)$ is the length of c. Let $\chi \in \mathrm{CR}(\mathfrak{S}_n)$ be the element $\chi(\sigma) = \mathrm{fix} \, G[\sigma]$. Then (2.35) is clearly equal to

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$p_\mu[\chi](\sigma)$, where μ is the cycle type of τ and the action of Λ on $\mathrm{CR}(\mathfrak{S}_n)$ is given by Definition 2.7.

So we see that $\mathrm{fix}_w(\tau, \sigma) = \mathrm{fix}\, F[\tau] p_\mu[\chi](\sigma) p_\lambda$, and thus that for $\sigma \in \mathfrak{S}_n$, the sum of the weights of $\mathfrak{S}_m$-orbits fixed by σ is:

$$
\mathrm{fix}_w \sigma = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \mathrm{fix}\, F[\tau] p_\mu[\chi](\sigma) p_\lambda
$$

where μ, λ are the cycle types of τ, σ respectively, $\qquad$ (2.36)

$$
= \sum_{\mu \vdash m} \frac{\mathrm{fix}\, F[\mu] p_\mu[\chi](\sigma)}{z_\mu} p_\lambda
$$

Summing over all m, we find that for the action of σ on $H(\mathbb{P})$, $\mathrm{fix}_w \sigma = Z_F[\chi](\sigma) p_\lambda$, and so the sum of the weights of the orbits under $\mathfrak{S}_n$ is:

$$
\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} Z_F[\chi](\sigma) p_\lambda = \sum_{\lambda \vdash n} Z_F[\chi](\sigma) \frac{p_\lambda}{z_\lambda} \qquad (2.37)
$$

where on the right-hand side, σ is any permutation of cycle type λ. Summing over all n gives the cycle index of $F \boxtimes G$, and by the definition of the inner plethysm of symmetric functions, we find that it is precisely $Z_F \boxtimes Z_G$. $\qquad \square$

2.4   The inner plethysm in Y

We recall that a 2-sorted species F is a functor $F : \mathbb{B}^2 \to \mathbb{B}$, with associated cycle index series Z_F:

$$
Z_F = \sum_{\lambda, \mu} \mathrm{fix}\, F[\lambda, \mu] \frac{p_\lambda(x)}{z_\lambda} \frac{p_\mu(y)}{z_\mu} \qquad (2.38)
$$

In direct analogy with Section 2.1.2, we define an R-*weighted, 2-sorted species* to be a functor $F : \mathbb{B}^2 \to \mathrm{Sets}_R$, and associate to it a weighted analytic functor $F() : \mathrm{Sets}_R{}^2 \to \mathrm{Sets}_R$:

$$
F(A, B) = \sum_{m, n \geq 0} F[m, n] \times A^m \times B^n / \mathfrak{S}_m \times \mathfrak{S}_n \qquad (2.39)
$$

where the action of $\mathfrak{S}_n \times \mathfrak{S}_m$ is $(\tau, \sigma) \cdot (t, f, g) = ((\tau, \sigma) \cdot t, f \circ \tau^{-1}, g \circ \sigma^{-1})$. We define the weight $w(t, f, g)$ to be $w(t) w(f) w(g)$, and associate to $F(A, B)$ its *type-series*,

$$
Z_{F(A, B)} = \sum_{\mathcal{O} \in F(A, B)} w(\mathcal{O}) \qquad (2.40)
$$

The relation between the cycle index and type-series is entirely analogous to that in the one-variable case. Let $R = \mathbb{Q}[[x_1, x_2, \ldots, y_1, y_2, \ldots]]$, let $\mathbb{P}_x$ denote
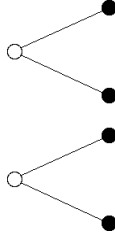
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Figure 2.1: The isomorphism class of G-structures

$\mathbb{P}$ with with weight function $w(i) = x_i$, and let $\mathbb{P}_y$ denote $\mathbb{P}$ with the weight function $w(i) = y_i$. Then,

$$Z_{F(\mathbb{P}_x, \mathbb{P}_y)} = Z_F \tag{2.41}$$

where Z_F on the right-hand side is the cycle index (2.4), considered as a formal series in the x_i and y_i. The proof is entirely analogous to that of (2.19).

An n-sorted species F, its cycle index, and its type-series, are defined in the obvious way, and the analog of (2.41) is easily verified.

So we see that, as in the one-variable case, we can consider the cycle index either as a symmetric function in the underlying variables, or as a series in the $p_i(x)$, $p_i(y)$, etc. We can also, however, consider it as a series in the $p_i(y)$ with coefficients in $\mathbb{Q}[[x_1, x_2, \dots]]$. This interpretation will prove useful.

2.4.1 Combinatorial description

In Section 2.4.2 we will construct an operation $\boxtimes_Y$, the *inner plethysm in* Y, which we describe here. Intuitively, $F \boxtimes_Y G$ is similar to the substitution $F(G(X, Y))$. Whereas an $F(G(X, Y))$-structure consists of an F-enriched set of G-structures, however, an $F \boxtimes_Y G$-structure can be said to consist of "an F-enriched set of G-structures which share the same Ys." We illustrate with an example.

Consider the case where $G = E_2(X \cdot E_2(Y))$. An isomorphism class (in fact, there is only one) of G-structures is shown in Figure 2.1. The white points are considered to be of sort X, the black of sort Y.

Let $F = E_2$. We would expect there to be 2 isomorphism classes of $F \boxtimes_Y G$-structures (see Figure 2.2), since an $F \boxtimes_Y G$-structure consists of a set of 2 G-structures which share the same Y-points. That this is, in fact, the case, we verify once we have set up the necessary machinery.

2.4.2 Construction

The standard notation for a 2-sorted species with underlying points of sorts X and Y is $F(X, Y)$. Since this is easily confused with our notation for the
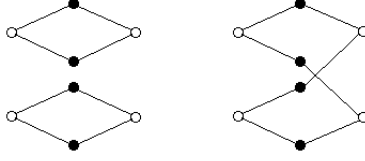
Figure 2.2: The isomorphism classes of $F \boxtimes_Y G$-structures

analytic functor associated to F, we will denote the analytic functor by F_{XY}, and make the following definition.

Definition 2.15. For a 2-sorted species $F(X, Y)$, we define

$$F_{X[Y]} : \text{Sets}_R \times \mathbb{B} \to \text{Sets}_R$$

$$F_{X[Y]}(A, B) = \sum_{n \geq 0} F[n, B] \times A^n / \mathfrak{S}_n \tag{2.42}$$

We will refer to $F_{X[Y]}$ as the *analytic functor in X* associated to F, or simply as a *partial analytic functor*. (The action of $\mathfrak{S}_n$ on $F[n, B] \times A^n$ in (2.42) is the obvious one: $\sigma \cdot (t, f) = (F[\sigma, 1_B](t), f \circ \sigma^{-1})$, where 1_B is the identity function on B.)

For any set $A \in \text{Sets}_R$, the analytic functor in X gives a functor $F_{X[Y]}(A, -) : \mathbb{B} \to \text{Sets}_R$, i.e., a weighted species, the underlying points of which are of sort Y. If $A = \mathbb{P}_x$, the cycle index of this species is a symmetric function in $y_1, y_2, \ldots$, with coefficients in $\mathbb{Q}[[x_1, x_2, \ldots]]$. Considered as a symmetric function in $x_1, x_2, \ldots, y_1, y_2, \ldots$, it is precisely the cycle index (2.4), as the following slightly more general result will show.

Lemma 2.16. *For the weighted, 1-sorted species $F_{X[Y]}(\mathbb{P}_x, -) : \mathbb{B} \to \text{Sets}_R$, we have:*

$$\text{fix}_w F_{X[Y]}(\mathbb{P}_x, \sigma) = \sum_\lambda \text{fix } F[\lambda, \mu] \frac{p_\lambda(x)}{z_\lambda} \tag{2.43}$$

where the sum is over all partitions λ, and μ is the cycle type of σ.

Proof. Let σ be an element of $\mathfrak{S}_n$. By (2.42), an element of $F_{X[Y]}(\mathbb{P}_x, n)$ is the orbit under $\mathfrak{S}_m$, for some m, of a pair (t, f), with $t \in F[m, n]$, $f : [m] \to \mathbb{P}_x$. The weight of the pair is $w(f) = f(1)f(2) \ldots$. Thus, we have an action of $\mathfrak{S}_m \times \mathfrak{S}_n$ on pairs (t, f):

$$(\tau, \sigma) \cdot (t, f) = (F[\tau, \sigma](t), f \circ \tau^{-1})$$

and $\text{fix}_w F_{X[Y]}(\mathbb{P}_x, \sigma)$ is the sum of the weights of $\mathfrak{S}_m$-orbits fixed by σ. Now, by the argument used to derive (2.16), the sum of the weights of functions f

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such that $f \circ \tau^{-1} = f$ is $p_\lambda(x)$, where λ is the cycle type of τ. So by Lemma 2.1, the sum of the weights of $\mathfrak{S}_m$-orbits fixed by σ is:

$$\frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \text{fix } F[\tau, \sigma] p_\lambda(x) = \sum_{\lambda \vdash m} \text{fix } F[\lambda, \mu] \frac{p_\lambda(x)}{z_\lambda}$$

where λ is the cycle type of τ, and μ is the cycle type of σ.

Summing over all m completes the proof. $\qquad\square$

Suppose now that F is a 1-sorted species, and $G(X, Y)$ is a 2-sorted species. We consider the composition $F(G_{X[Y]}(-, -)) : \text{Sets}_R \times \mathbb{B} \to \text{Sets}_R$.

Lemma 2.17. *With F and G as above, there exists a 2-sorted species $H(X, Y)$ such that $H_{X[Y]} = F(G_{X[Y]}(-, -))$.*

Proof. We construct H as follows. For finite sets U, V, we define an element of $H[U, V]$ to be:

(1) A partition $U_1 + \cdots + U_n = U$

(2) G-structures $\alpha_1, \ldots, \alpha_n$, with $\alpha_i \in G[U_i, V]$

(3) An F-structure $\alpha \in F[\{\alpha_1, \ldots, \alpha_n\}]$

For $A \in \text{Sets}_R$, $B \in \mathbb{B}$, an element of $H_{X[Y]}(A, B)$ consists of the orbit under $\mathfrak{S}_n$, for some n, of a pair (t, f), with $t \in H[n, B]$, and $f : [n] \to A$.

Now an element of $F(G_{X[Y]}(A, B))$ consists of the orbit under $\mathfrak{S}_n$, for some n, of a pair (t', f'), with $t' \in F[n]$, $f' : [n] \to G_{X[Y]}(A, B)$. Thus, for $i \in [n]$, $f'(i)$ consists of the orbit under $\mathfrak{S}_{m_i}$, for some m_i, of a pair (t_i', f_i'), where $t_i' \in G[m_i, B]$, $f_i' : [m_i] \to A$.

We can thus map the orbit of (t, f) to the orbit of an element (t', f') as follows. Let $\gamma : \{\alpha_1, \ldots, \alpha_n\} \to [n]$ be any bijection, and let $\overline{\sigma} \in \mathfrak{S}_n$ be the permutation defined by $\overline{\sigma}(i) = k \Leftrightarrow \gamma(\alpha_k) = i$. For $1 \le i \le n$, let $m_i = |U_i|$, and let $\gamma_i : U_i \to [m_i]$ also be bijections. Define $t' = F[\gamma](\alpha)$, and for $1 \le i \le n$, let $f' : [n] \to G_{X[Y]}(A, B)$ be defined by $f'(i) = \mathcal{O}_{\mathfrak{S}_{m_k}}(G[\gamma_k, 1_B](\alpha_k), f \circ \gamma_k^{-1})$ for $1 \le i \le n$, where $k = \overline{\sigma}(i)$, and 1_B is the identity function on B.

The verification that this gives a natural isomorphism between functors is straightforward, and entirely analogous to the proof of Lemma 2.5. $\qquad\square$

Definition 2.18. For species F, G as above, we define the *inner plethysm in Y* of F and G to be the species H constructed in the proof of Lemma 2.17, and denote it $F \boxtimes_Y G$.

In order to calculate the cycle index of $F \boxtimes_Y G$, we first give a slight generalization of the λ-ring of central functions of a group.

Definition 2.19. For a group G and a λ-ring R, we define the ring $\text{CF}_R(G)$ of central functions from G to R to be the set of all functions from G to R which

are constant on conjugacy classes. The Adams operations on $\mathrm{CF}_R(G)$ are given by:

$$(\psi^k(f))(g) = \psi_R^k(f(g^k)) \tag{2.44}$$

for all $f \in \mathrm{CF}_R(G)$, $g \in G$. Here ψ_R^k denotes the kth Adams operation of the λ-ring R.

By Lemma 2.16, in order to calculate the desired cycle index, it will be sufficient to calculate $\mathrm{fix}_w (F \boxtimes_Y G)(\mathbb{P}_x, \sigma)$ for a permutation σ.

Lemma 2.20. *For species F and G as above, define $\chi : \mathfrak{S}_n \to \Lambda_x$ by:*

$$\chi(\sigma) = \mathrm{fix}_w G_{X[Y]}(\mathbb{P}_x, \sigma) = \sum_\lambda \mathrm{fix}\, G[\lambda, \mu] \frac{p_\lambda(x)}{z_\lambda}$$

where μ is the cycle type of $\sigma \in \mathfrak{S}_n$. Then:

$$\mathrm{fix}_w (F \boxtimes_Y G)(\mathbb{P}_x, \sigma) = Z_F[\chi](\sigma) \tag{2.45}$$

where the action of Λ on $\mathrm{CF}_{\Lambda_x}(\mathfrak{S}_n)$ is given by Definition 2.7.

Proof. By definition, an element of $(F \boxtimes_Y G)(\mathbb{P}_x, n)$ is the orbit under $\mathfrak{S}_m$, for some m, of a pair (t, f), with $t \in F[m]$, $f : [m] \to G_{X[Y]}(\mathbb{P}_x, n)$. The symmetric group $\mathfrak{S}_n$ acts on $G_{X[Y]}(\mathbb{P}_x, n)$, and so we have an action of $\mathfrak{S}_m \times \mathfrak{S}_n$ on the set of pairs (t, f). We wish to calculate $\mathrm{fix}_w (F \boxtimes_Y G)(\mathbb{P}_x, \sigma)$, the sum of the weights of $\mathfrak{S}_m$-orbits fixed by σ.

Suppose a pair $(\tau, \sigma) \in \mathfrak{S}_m \times \mathfrak{S}_n$ fixes a pair (t, f). Then τ must fix t (i.e., $F[\tau](t) = t$), and f must satisfy

$$G_{X[Y]}(\mathbb{P}_x, \sigma) \circ f \circ \tau^{-1} = f \tag{2.46}$$

Consider a cycle c of τ of length l, with $i \in c$. The image of any element of c under f is determined by the image of i, and all such images have the same weight, so the cycle c contributes a factor of $w(f(i))^l$ to the weight of f.

Since c has length l, (2.46) implies that $G_{X[Y]}(\mathbb{P}_x, \sigma^l)$ must fix $f(i)$. Thus, $\mathrm{fix}_w G_{X[Y]}(\mathbb{P}_x, \sigma^l)$ counts by weight all possible choices for $f(i)$. Since $f(i)$ is an element of $G_{X[Y]}(\mathbb{P}_x, n)$, $w(f(i))$ is a monomial in the variables $x_1, x_2, \ldots$, and $w(f(i))^l$ is obtained from $w(f(i))$ by replacing each variable x_j with $x_j{}^l$.

Replacing each x_j with $x_j{}^l$ in a symmetric function g amounts to calculating the plethysm $p_l \circ g$. So we see that the sum of the weights of functions f which satisfy (2.46) is:

$$\prod_c p_{l(c)} \circ \mathrm{fix}_w G_{X[Y]}(\mathbb{P}_x, \sigma^{l(c)}) = p_\lambda[\chi](\sigma)$$

where the product is over all cycles c of τ, and λ is the cycle type τ.

Thus, by Lemma 2.1, the sum of the weights of $\mathfrak{S}_m$-orbits fixed by σ is:

$$\frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \mathrm{fix}_w(\tau, \sigma) = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \mathrm{fix}\, F[\tau] p_\lambda[\chi](\sigma)$$

$$= \sum_{\lambda \vdash m} \mathrm{fix}\, F[\lambda] \frac{p_\lambda[\chi](\sigma)}{z_\lambda}$$

Summing over all m completes the proof. $\qquad\qquad\qquad$ $\square$

Lemma 2.20 provides a method for calculating the cycle index of $F \boxtimes_Y G$, which can be described succinctly by defining an operation $\boxtimes_Y : \Lambda \times \Lambda_{xy} \to \Lambda_{xy}$ as follows. Suppose $f \in \Lambda$, and $g \in \Lambda_{xy}$ is homogeneous of degree n in y, that is,

$$g = \sum_{\mu \vdash n} a_\mu(x) \frac{p_\mu(y)}{z_\mu}$$

where $a_\mu \in \Lambda_x$. Define $\chi : \mathfrak{S}_m \to \Lambda_x$ by $\chi(\sigma) = a_\mu(x)$, where μ is the cycle type of σ. Then $\chi \in \mathrm{CF}_{\Lambda_x}(\mathfrak{S}_n)$, and we can define:

$$f \boxtimes_Y g = \sum_{\mu \vdash n} f[\chi](\sigma) \frac{p_\mu(y)}{z_\mu}$$

where σ has cycle type μ, and the action of Λ on $\mathrm{CF}_{\Lambda_x}(\mathfrak{S}_n)$ is given by Definition 2.7. (The λ-ring structure of $\mathrm{CF}_{\Lambda_x}(\mathfrak{S}_n)$ is given by Definition 2.19.) We extend $\boxtimes_Y$ to a mapping $\Lambda \times \Lambda_{xy} \to \Lambda_{xy}$ by linearity. Lemma 2.20 can then be expressed as:

$$Z_{F \boxtimes_Y G} = Z_F \boxtimes_Y Z_G \qquad\qquad (2.47)$$

We define operations $\boxtimes_X$, $\boxtimes_{XY}$, $\boxtimes_{XZ}$, and so forth, in the obvious way. To define $\boxtimes_{XZ}$, for example, we define $G_{[X]Y[Z]}$ for a 3-sorted species G by:

$$G_{[X]Y[Z]}(A, B, C) = \sum_{n \geq 0} G[A, n, C] \times B^n / \mathfrak{S}_n$$

and for a 1-sorted species F, we define $F \boxtimes_{XZ} G$ to be the species H such that $H_{[X]Y[Z]} = F(G_{[X]Y[Z]}(-, -, -))$. The analogs of Lemmas 2.16, 2.17, and 2.20, are easily verified.

We now return to the example given in Section 2.4.1. We have $Z_F = h_2 = \frac{1}{2}(p_1{}^2 + p_2)$, and,

$$\begin{aligned}
Z_G &= h_2 \circ (p_1(x) h_2(y)) \\
&= \frac{1}{8} p_1(y)^4 p_1(x)^2 + \frac{1}{4} p_1(y)^2 p_1(x)^2 p_2(y) + \frac{1}{8} p_1(x)^2 p_2(y)^2 + \\
&\quad \frac{1}{4} p_2(y)^2 p_2(x) + \frac{1}{4} p_4(y) p_2(x)
\end{aligned}$$

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Thus, by Lemma 2.20,

$$Z_F \boxtimes_Y Z_G = \frac{1}{4} p_4(y) p_2(x)^2 + \frac{1}{4} p_4(y) p_4(x) + \frac{3}{8} p_2(y) p_1(y)^2 p_2(x)^2 +$$

$$\frac{1}{8} p_2(y) p_1(y)^2 p_1(x)^4 + \frac{7}{16} p_2(y)^2 p_2(x)^2 + \frac{1}{16} p_2(y)^2 p_1(x)^4 +$$

$$\frac{1}{4} p_2(y)^2 p_1(x)^2 p_2(x) + \frac{1}{16} p_1(y)^4 p_2(x)^2 + \frac{3}{16} p_1(y)^4 p_1(x)^4$$

(This calculation was performed with the aid of Maple.)

Setting $p_i(x) = x^i$, $p_i(y) = y^i$ for all i in $Z_F \boxtimes_Y Z_G$ yields $2x^4 y^4$, indicating that there are 2 unlabeled $F \boxtimes_Y G$-structures on 4 points of sort X and 4 points of sort Y, as expected.

2.5   Polynomial maps

We denote by $\mathbf{S}_n$ the set of all n-sorted species, and by $\mathfrak{M}_n$ the set of all n-sorted molecular species. We consider two species A and B to be equal, and write "$A = B$," if they are naturally isomorphic. The set $\mathbf{S}_n$ is a half-ring, and not a ring, since it has operations $+$ and $\cdot$, but no operation of subtraction. Just as the integers can be constructed from the natural numbers, the ring $\mathbf{V}_n$ of all n-sorted *virtual species* can be constructed from $\mathbf{S}_n$. As shown in [15], we have the following half-ring and ring isomorphisms:

$$\mathbf{S}_n \cong \mathbb{N}[[\mathfrak{M}_n]]$$
$$\mathbf{V}_n \cong \mathbb{Z}[[\mathfrak{M}_n]]$$

Similarly, by $\mathbf{S}_n^*$ $\mathbf{V}_n^*$ we denote the sets of strictly finite species and virtual species, respectively. We have (again, see [15]):

$$\mathbf{S}_n^* \cong \mathbb{N}[\mathfrak{M}_n]$$
$$\mathbf{V}_n^* \cong \mathbb{Z}[\mathfrak{M}_n]$$

The operations $+$ and $\cdot$ extend from $\mathbf{S}_n$ to $\mathbf{V}_n$ by construction. In this section we develop techniques—essentially those used in [15]—to extend others, such as $\circ$, $\boxtimes$, $\boxtimes_Y$, and to prove that the corresponding identities for symmetric functions ((2.23), (2.29), and (2.47), in the case of these three) remain valid for virtual species.

2.5.1   Polynomial maps on species

We will use the notation $(a_i)_{i \in I}$ to denote an indexed collection of (not-necessarily-distinct) objects a_i (indexed by the elements of a some set I). Strictly speaking,

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$(a_i)_{i \in I}$ is shorthand for the function defined on I which has the value a_i at i for every $i \in I$. Thus, $(a_i)_{i \in \mathbb{N}}$, with $a_i \in \mathbb{R}$, denotes a sequence of real numbers, and if R is an n-sorted species with molecular series $\sum_{N \in \mathfrak{M}_n} a_N N$, then $(a_N)_{N \in \mathfrak{M}_n}$ is the collection of coefficients of R.

Given a species R, $[R]$ will denote the collection of coefficients of the molecular series of R.

We will often be concerned with functions from species to species. A function $f : \mathbf{S}_n \to \mathbf{S}_m$ determines a collection of functions $(f_M)_{M \in \mathfrak{M}_m}$ via its action on molecular series:

$$f(R) = \sum_{M \in \mathfrak{M}_m} f_M([R])M \tag{2.48}$$

Definition 2.21. If the functions f_M in (2.48) are polynomials in the coefficients $[R] = (a_N)_{N \in \mathfrak{M}_n}$ (that is, if f_M can be written as a polynomial in $(a_N)_{N \in S}$ for some finite subset S of $\mathfrak{M}_n$), then f is a *polynomial map on species*, or simply a *polynomial map*.

More generally, let I be a finite index set and suppose that f is a function from $\prod_{i \in I} \mathbf{S}_{n_i}$ to $\mathbf{S}_m$, where the n_i are positive integers. Then, as above, f determines functions $(f_M)_{M \in \mathfrak{M}_m}$. If $f_M(([R_i])_{i \in I})$ is a polynomial in the coefficients $([R_i])_{i \in I}$, we will say that f is a polynomial map on species in this case as well. A map $f : \prod_{i \in I} \mathbf{S}_{n_i} \to \prod_{j \in J} \mathbf{S}_{m_j}$ will be considered a polynomial map on species if each component function is a polynomial map on species.

Remark. Any occurrence of $\mathbf{S}_i$ in the above definitions may be replaced by $\mathbf{S}_i^*$, or any other subset of $\mathbf{S}_i$; the resulting maps will also be considered *polynomial maps*.

The chief significance of polynomial maps is that a polynomial map f on species can be extended to a polynomial map on virtual species simply by allowing the coefficients in the molecular series upon which f operates to take on negative values. Since two polynomials which agree for all positive values of their arguments must be identically equal, this extension is the unique polynomial extension of f to virtual species. Given a polynomial operation, we will henceforth take this extension for granted and use it without comment.

We now demonstrate that certain maps on species are polynomial.

Definition 2.22. A binary operation $*$ from $\mathbf{S}_n \times \mathbf{S}_n$ to $\mathbf{S}_n$ is *bilinear* if, for all $A_i, B_i, C \in \mathbf{S}_n$,

(1) $(A_1 + A_2 + \dots) * C = A_1 * C + A_2 * C + \dots$

(2) $A * (B_1 + B_2 + \dots) = A * B_1 + A * B_2 + \dots$

whenever the sums $A_1 + A_2 + \dots$, $B_1 + B_2 + \dots$, respectively, are defined.

We define *linearity* (of a map from $\mathbf{S}_n$ to $\mathbf{S}_n$) entirely analogously.

Definition 2.23. A binary operation $*$ from $\mathbf{S}_n \times \mathbf{S}_n$ to $\mathbf{S}_n$ is a *polynomial operation*, or simply *polynomial*, if the map $(A, B) \mapsto A * B$ is a polynomial map on species.

We note that $A * B$, where A and B are virtual species, is well-defined for any polynomial binary operation $*$.

If f and g are polynomial maps, then it is immediate that their composition, when defined, is also a polynomial map.

As in [15], we define a species A to be a subspecies of the species B if $A[U] \subseteq B[U]$ for all finite sets U, and the inclusion is a natural transformation. If A is a molecular species, this is equivalent to saying that A occurs in the molecular series of B. We note that $* : \mathbf{S}_n \times \mathbf{S}_n \to \mathbf{S}_n$ is bilinear, then a given molecular species T can be a subspecies of $M_1 * M_2$ for only finitely many pairs $(M_1, M_2) \in \mathfrak{M}_n \times \mathfrak{M}_n$; otherwise,

$$\left(\sum_{M_1 \in \mathfrak{M}_n} M_1 \right) * \left(\sum_{M_2 \in \mathfrak{M}_n} M_2 \right)$$

would not be defined. This observation also gives:

Lemma 2.24. *If* $* : \mathbf{S}_n \times \mathbf{S}_n \to \mathbf{S}_n$ *is bilinear, then it is a polynomial operation.*

Proof. Let $A = \sum_{M \in \mathfrak{M}_n} a_M M$ and $B = \sum_{M \in \mathfrak{M}_n} b_M M$ be the molecular series of species A and B. Then,

$$A * B = \sum_{M_1, M_2 \in \mathfrak{M}_n} a_{M_1} b_{M_2} M_1 * M_2 \tag{2.49}$$

by bilinearity. Since a given molecular species T can be a subspecies of $M_1 * M_2$ for only finitely many pairs $(M_1, M_2) \in \mathfrak{M}_n \times \mathfrak{M}_n$, we can collect terms on the right-hand side of (2.49), yielding a molecular series whose coefficients are polynomials in the a_{M_1} and b_{M_2}. $\square$

Remark. Similarly, any linear operation is polynomial. Thus, for example, the diagonal map $\nabla : \mathbf{S}_2 \to \mathbf{S}_1$ (defined by $(\nabla F)[U] = F[U, U]$), and the map $A \mapsto A'$ from $\mathbf{S}_1$ to $\mathbf{S}_1$, are polynomial.

Lemma 2.25. *The following binary operations are polynomial:* $+$, $\cdot$, $\times$, $\circ$.

Proof. For $+$ this is immediate, since the coefficient of $M \in \mathfrak{M}_n$ in the molecular series of $A + B$ is the sum of the coefficients of M in the molecular series of A and B. For $\cdot$ and $\times$, which are bilinear, the lemma follows immediately from Lemma 2.24. For $\circ$, which is not bilinear, see [15]. $\square$

The Cartesian product in Y is clearly bilinear, and thus polynomial.

Lemma 2.26. *The map* $F(X, Y) \mapsto F(X, 1)$ *is polynomial.*

Proof. It is sufficient to observe that given a 2-sorted molecular species $M(X, Y)$, $M(X, 1)$ is a finite sum of 1-sorted molecular species. $\qquad\square$

In order to show that the map Φ introduced in Section 2.3.1 is polynomial, we first show that it has a certain multiplicative property:

Lemma 2.27. *For species F_1 and F_2,*

$$\Phi(F_1 + F_2) = \Phi(F_1) \times_Y \Phi(F_2)$$

Proof. An element $f \in \Phi(F_1 + F_2)[U, V]$ is a function from U to the disjoint union $F_1[V] \cup F_2[V]$. Letting $U_i = f^{-1}(F_i[V])$ for $i = 1, 2$, we obtain functions $f_i : U_i \to F_i[V]$, with U the disjoint union of U_1 and U_2 (f_i is simply $f|_{U_i}$). This gives a natural bijection from $\Phi(F_1 + F_2)[U, V]$ to $\sum_{U_1 + U_2 = U} \Phi(F_1)[U_1, V] \times \Phi(F_2)[U_2, V]$, proving the claim. $\qquad\square$

Lemma 2.28. $\Phi : \mathbf{S}_1 \to \mathbf{S}_2$ *is a polynomial map on species.*

Proof. For $A \in \mathbf{S}_2$ and $i \in \mathbb{N}$, let $A^{\times_Y i}$ denote $A \times_Y A \times_Y A \times_Y \ldots$ (i factors). We take $A^{\times_Y 0}$ to be 1.

By Lemma 2.27, we have, for any $M \in \mathfrak{M}_1$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\Phi(nM) &= \Phi(M)^{\times_Y n} \\
&= (E(Y) + \Phi(M) - E(Y)) \\
&= \sum_{i=0}^{n} \binom{n}{i} (\Phi(M) - E(Y))^{\times_Y i}
\end{aligned} \tag{2.50}$$

Thus, for any molecular species $M_2 \in \mathfrak{M}_2$, the coefficient of M_2 in the molecular series of $\Phi(nM)$ is a polynomial in n. Applying Lemma 2.27 again, and the fact that $\times_Y$ is a polynomial operation, we have that the coefficient of M_2 in the molecular series of $\Phi(a_1 N_1 + a_2 N_2 + \ldots)$ is a polynomial in $a_1, a_2, \ldots$, provided that $a_1 N_1 + a_2 N_2 + \ldots$ is a finite sum.

Now consider the coefficient of $M_2 \in \mathfrak{M}_2$ in the molecular series of

$$\Phi\left(\sum_{M \in \mathfrak{M}_1} a_M M \right)$$

Since M_2 is molecular, it must be homogeneous, say of degree (n, m). By the definition of Φ, $M_2[n, m]$ thus consists of the orbit, under $\mathfrak{S}_n \times \mathfrak{S}_m$, of some function

$$f : [n] \to \left(\sum_{M \in \mathfrak{M}_1} a_M M \right)[m]$$

Thus, we see that the coefficient of M_2 depends only upon those molecular species $M \in \mathfrak{M}_1$ which are homogeneous of degree m—in fact, if S is the set

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of such molecular species, the coefficient of M_2 in $\Phi(\sum_{M \in \mathfrak{M}_1} a_M M)$ is equal to the coefficient of M_2 in $\Phi(\sum_{M \in S} a_M M)$. Since this latter is a finite sum, the lemma is proven. $\square$

Lemma 2.29. *Lemma 2.27 holds when F_1 and F_2 are virtual species.*

Proof. The maps $(F_1, F_2) \mapsto \Phi(F_1 + F_2)$ and $(F_1, F_2) \mapsto \Phi(F_1) \times_Y \Phi(F_2)$ are both polynomial maps on species, and by Lemma 2.27, they agree when F_1, F_2 are species. Therefore, they agree on virtual species. $\square$

Lemma 2.30. *The map $\boxtimes_Y : \mathbf{S}_1^* \times \mathbf{S}_2 \to \mathbf{S}_2$ is a polynomial map on species.*

Proof. Let $F = \sum_{M \in \mathfrak{M}_1} a_M M$ and $G = \sum_{N \in \mathfrak{M}_2} b_N N$ be the molecular series of F and G, and suppose R is a molecular subspecies of $F \boxtimes_Y G$. We must show that the coefficient of R in the molecular series of $F \boxtimes_Y G$ is a polynomial in the a_M and b_N.

Since R is molecular, it must be homogeneous of degree (m_1, m_2), for some m_1, m_2. An element of R consists of:

(1) A partition $U_1 + \cdots + U_n = [m_1]$

(2) G-structures $\alpha_1, \ldots, \alpha_n$, with $\alpha_i \in G[U_i, m_2]$

(3) An F-structure $\alpha \in F[\{\alpha_1, \ldots, \alpha_n\}]$

Thus,

$$\alpha_i \in \sum_{N \in \mathfrak{M}_2} b_N N[U_i, m_2] \tag{2.51}$$

for $1 \leq i \leq n$, where in the sum on the right, $b_N N[U_i, m_2]$ denotes the disjoint union of b_N copies of the set $N[U_i, m_2]$. (Up to the action of $\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}$, this is the only element of R, since R is molecular.)

We will think of the b_N copies of the set $N[U_i, m_2]$ as each being colored in one of b_N distinct colors. In (2.51), α_i must be an element of exactly one copy of $N[U_i, m_2]$, for some $N \in \mathfrak{M}_2$. We will say that α_i *occurs* in this copy, and that α_i occurs in the molecular species N.

Let $N_1, \ldots, N_s$ be the distinct molecular species in which α_i occurs (for some i), and for each i, let k_i be the number of copies of N_i in which some α_j appears. Let M_1 be the molecular subspecies of F in which α occurs. Let $a_1 = a_{M_1}$, and $b_i = b_{N_i}$ for $1 \leq i \leq s$. Define $F_0 = M_1$, $G_0 = \sum_{i=1}^s k_i N_i$. Then $F_0 \boxtimes_Y G_0$ is naturally isomorphic to a subspecies of $F \boxtimes_Y G$, and in fact, for each choice of k_i out of b_i copies of N_i, and one copy out of a_1 of M_1, we obtain a subspecies of $F \boxtimes_Y G$ which is isomorphic to $F_0 \boxtimes_Y G_0$. Let p be the coefficient of R in $F_0 \boxtimes_Y G_0$. Then we obtain the following contribution to the coefficient of R in the molecular series of $F \boxtimes_Y G$:

$$p a_1 \prod_{i=1}^s \binom{b_i}{k_i} \tag{2.52}$$

To show that the coefficient of R is in fact a finite sum of such terms, we proceed as follows. Suppose R^* is another molecular subspecies of $F \boxtimes_Y G$ which is isomorphic to R. Then R^* must be homogeneous of degree (m_1, m_2), an element of R^* consists of

(1) A partition $U_1^* + \cdots + U_{n^*}^* = [m_1]$

(2) G-structures $\alpha_1^*, \ldots, \alpha_{n^*}^*$, with $\alpha_i^* \in G[U_i^*, m_2]$

(3) An F-structure $\alpha^* \in F[\{\alpha_1^*, \ldots, \alpha_{n^*}^*\}]$

and we can define $N_1^*, \ldots, N_{s^*}^*$, k_i^*, a_1^*, b_i^*, M_1^*, F_0^*, G_0^*, and p^* just as we did their non-starred analogues. If $F_0^* \boxtimes_Y G_0^* = F_0 \boxtimes_Y G_0$ (that is, if they are isomorphic), then R^* is counted in the sum (2.52). If not, we obtain a new contribution to the coefficient of R, equal to:

$$p^* a_1^* \prod_{i=1}^{s^*} \binom{b_i^*}{k_i^*}$$

There are only finitely many possibilities for the species $F_0^* \boxtimes_Y G_0^*$, which can be seen as follows. Since F is strictly finite, there are only finitely many possibilities for F_0. And, each N_i^* must be homogeneous of a degree (d, m_2), with $d \leq m_1$, so there are only finitely many possibilities for the N_i^*. Finally, the k_i^* are bounded by coefficients in the molecular series of G. $\qquad \square$

2.5.2 Polynomial maps on symmetric functions

A map $f : \Lambda \to \Lambda$ determines a function f_λ for every partition λ:

$$f\left(\sum_{\lambda \in \mathcal{P}} a_\lambda \frac{p_\lambda}{z_\lambda}\right) = \sum_{\lambda \in \mathcal{P}} f_\lambda((a_\mu)_{\mu \in \mathcal{P}}) \frac{p_\lambda}{z_\lambda} \tag{2.53}$$

Definition 2.31. If the functions f_λ in (2.53) are polynomials in the a_μ, then f is a *polynomial map on symmetric functions*, or simply a *polynomial map*.

Similarly, a map $f : \mathbf{S}_m \to \Lambda$ determines functions f_λ:

$$f(R) = \sum_{\lambda \in \mathcal{P}} f_\lambda([R]) \frac{p_\lambda}{z_\lambda} \tag{2.54}$$

Definition 2.32. If the functions f_λ in (2.54) are polynomials in the coefficients $[R]$, then f is a *polynomial map from species to symmetric functions*, or simply a *polynomial map*.

We extend this definition to include polynomial maps from $\mathbf{S}_m \times \mathbf{S}_n \to \Lambda_{xy}$, etc., in the obvious way.

Lemma 2.33. *The map $F \mapsto Z_F$ is a polynomial map from species to symmetric functions.*

Proof. For simplicity, we consider the case where F is 1-sorted, with molecular series $F = \sum_{M \in \mathfrak{M}_1} a_M M$. Then,

$$Z_F = \sum_{M \in \mathfrak{M}_n} a_M Z_M \tag{2.55}$$

The coefficient of $\frac{p_\lambda}{z_\lambda}$ in Z_M, for some partition λ, can be non-zero for only finitely many Z_M (since there are only finitely many molecular species homogeneous of a given degree), and the lemma follows immediately. $\square$

We now have the tools to verify that equalities such as (2.23) hold for virtual species. To show that $Z_{F \circ G} = Z_F \circ Z_G$, we observe that the maps $(F, G) \mapsto Z_{F \circ G}$ and $(F, G) \mapsto Z_F \circ Z_G$ are both polynomial maps from species to symmetric functions. By (2.23), they agree on species; therefore, they agree on virtual species.

Chapter 3

Digraphs

3.1 G-digraphs

For a 1-sorted species G, we define a G-digraph on a set U to be a digraph $\mathcal{D}$ with vertex set U, together with a G-structure on the set of arcs out of each vertex. We denote by D_G the species of G-digraphs in which loops are not allowed, and by D_G° the species of G-digraphs in which they are.

Lemma 3.1. $D_G^\circ = \nabla\Phi(E \cdot G)$, where Φ is the map defined in Section 2.3.1.

Proof. For a finite set V, an element of $(E \cdot G)[V]$ consists of a subset of V, together with a G-structure on that subset. By the definition of Φ, an element of $\Phi(E \cdot G)[U, V]$ consists of a function from U to $(E \cdot G)[V]$—i.e., it associates to each element of U a subset of V and a G-structure on that subset.

An element of $\nabla\Phi(E \cdot G)[U]$ thus associates to each $u \in U$ a subset of U, and a G-structure on that subset. This is equivalent to specifying a G-digraph on U. $\qquad\square$

Lemmas 3.1 and 2.10 allow us to calculate the cycle index of D_G° for any species G. If $G = E$, for example, D_G° is the species of all digraphs (with loops allowed), and we find that the isomorphism-types generating function for D_G° is:

$$\widetilde{D_E^\circ}(x) = 2x + 10x^2 + 104x^3 + 3044x^4 + 291968x^5 + 96928992x^6 + \ldots$$

(Counting such digraphs is equivalent to counting relations on a set; see Section 5.1 of [5].)

3.2 Removing loops

In order to deal with D_G, we have the following lemma:

Lemma 3.2. *If G is a species, then $D_{G+G'} = D_G^\circ$.*

Proof. A G-structure on the edges out of a vertex with a loop can also be thought of as a G'-structure on these edges with the loop removed. So given a D_G°-structure on a set U, we can remove all the loops and replace the G-structures at those vertices with the same structures considered as G'-structures on one fewer objects. This gives a natural bijection from $D_G^\circ[U]$ to $D_{G+G'}[U]$. $\qquad\square$

Thus, a way to calculate the cycle index series of D_G for some species G is to find G_1 such that $G_1 + G_1{}' = G$ and calculate the cycle index series of $D_{G_1}^\circ$. We note that one such solution (provided the sum converges) is $G_1 = G - G' + G'' - G''' + \ldots$; the G_1 so obtained may therefore be a virtual species. We need to extend the definition of D_G° and D_G to virtual species and show that Lemma 3.2 remains true when G is a virtual species.

Lemma 3.1, combined with Lemma 2.28, allows us to conclude that the map $G \mapsto D_G^\circ$ is a polynomial map on species. Therefore D_G° is well-defined for virtual species.

Lemma 3.3. *The map $G \mapsto D_G$ is a polynomial map on species.*

Proof. Let M be a molecular subspecies of D_G, where $G = \sum_{N \in \mathfrak{M}_1} a_N N$. Suppose M is homogeneous of degree m. An element of $M[m]$ consists of a digraph $\mathcal{D}$ on $[m]$, together with a G-structure on the edges out of every vertex. Any G-structure is an N-structure, for some $N \in \mathfrak{M}_1$, colored in one of a_N colors.

Let $N_1, \ldots N_s$, be the elements of $\mathfrak{M}_1$ whose structures appear at the vertices of $\mathcal{D}$. Furthermore, let k_i be the number of distinct colors of N_i-structures which appear, and let $a_i = a_{N_i}$.

We give an example of such a molecular subspecies in Figure 3.1. Here $m = 6$, $s = 2$, and $G = a_{N_1} N_1 + a_{N_2} N_2 + \ldots$ for two molecular species N_1 and N_2 (homogeneous of degrees 1 and 3, respectively). We see from the figure that $k_1 = 3, k_2 = 1$.

Let $T = \sum_{i=1}^s k_i N_i$. Then D_T is naturally isomorphic to a subspecies of D_G, and any choice of k_i colors out of a_i yields a subspecies of D_G which is isomorphic to D_T. (Switching red, green, and blue to yellow, black, and white in Figure 3.1, for example, yields a distinct molecular subspecies isomorphic to the first.) Let p be the coefficient of M in the molecular series of D_T. Then we have the following contribution to the coefficient of M in the molecular series of D_G:

$$p \prod_{i=1}^s \binom{a_i}{k_i} \tag{3.1}$$

We will now show that the coefficient of M in D_G is in fact a sum of such terms.

Suppose M^* is another molecular subspecies of D_G which is isomorphic to M. Then M^* must be homogeneous of degree m. We can define s^*, $N_1^*, \ldots, N_{s^*}^*$, a_i^*, and k_i^* just as we defined their analogues for M, and let $T^* = \sum_{i=1}^{s^*} k_i^* N_i^*$. If $T^* = T$ then M^* is counted in (3.1). If not, let p^* be the coefficient of M^*
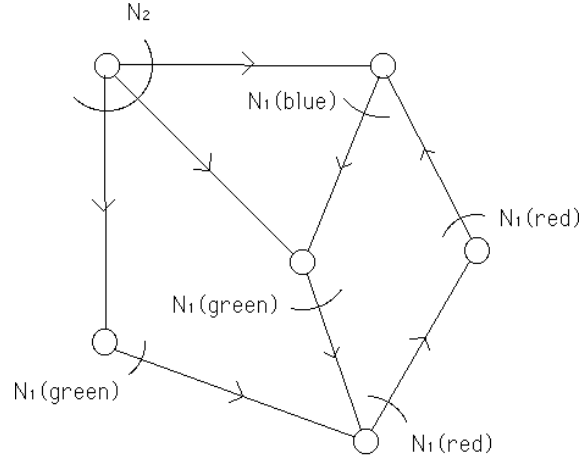
Figure 3.1: A molecular subspecies

in T^*; we obtain a new expression of the same form as (3.1), with all quantities replaced by their starred analogues.

To see that there are only finitely many possibilities for the species T^*, we note that any molecular species N_i^* occurring in the molecular series of T^* must be homogeneous of some degree less than m (since a vertex in a loopless digraph on m vertices can have arcs to at most $m - 1$ vertices). $\qquad\square$

Lemma 3.4. *If G is a virtual species, then $D_G = D_{G+G'}^\circ$.*

Proof. We recall that the map $A \mapsto A'$ is polynomial (since it is linear). By our observation that $G \mapsto D_G^\circ$ is polynomial, we conclude that $G \mapsto D_{G+G'}^\circ$ is polynomial as well. We have just seen that $G \mapsto D_G$ is polynomial, and by Lemma 3.2, the two maps agree on species. Therefore, they agree on virtual species. $\qquad\square$

It remains to calculate the cycle index series of D_G when G is a virtual species. In order to do this, we will make use of Lemma 2.29, which shows that for any species F, $\Phi(F - F) = \Phi(F) \times_Y \Phi(-F)$. Since $\Phi(0) = E(Y)$, this gives a way to calculate the cycle index series of $\Phi(-F)$. In fact, applying (2.28), we see that

$$Z_{\Phi(-F)} = \sum_\mu \exp\left(-\sum_{i \geq 1} \frac{\operatorname{fix} F[\sigma^i] p_i(x)}{i}\right) \frac{p_\mu(y)}{z_\mu} \qquad (3.2)$$

where σ is any permutation of cycle type μ.

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Combining Lemma 2.29 and (3.2) gives a general expression for $Z_{\Phi(F_1-F_2)}$ for any species F_1 and F_2:

$$
\begin{aligned}
Z_{\Phi(F_1-F_2)} &= \sum_\mu \exp\left(\sum_{i\geq 1} \operatorname{fix} F_1[\sigma^i]\frac{p_i(x)}{i}\right) \exp\left(-\sum_{i\geq 1} \operatorname{fix} F_2[\sigma^i]\frac{p_i(x)}{i}\right)\frac{p_\mu(y)}{z_\mu} \\
&= \sum_\mu \exp\left(\sum_{i\geq 1} \left(\operatorname{fix} F_1[\sigma^i] - \operatorname{fix} F_2[\sigma^i]\right)\frac{p_i(x)}{i}\right)\frac{p_\mu(y)}{z_\mu} \\
&= \sum_{\lambda,\mu}\left(\prod_{k\geq 1}\left(\operatorname{fix} F_1[\sigma^k] - \operatorname{fix} F_2[\sigma^k]\right)^{\beta_k}\right)\frac{p_\lambda(x)}{z_\lambda}\frac{p_\mu(y)}{z_\mu}
\end{aligned}
$$

$$(3.3)$$

where σ denotes a permutation of cycle type μ, and β_k is the number of k-cycles in a permutation of cycle type λ.

3.3 Applications

3.3.1 All digraphs

The species of all digraphs is D_E, where E is the species of sets. We saw in Section 3.2 that to calculate the cycle index of this species, we must find a (possibly virtual) species G such that $G + G' = E$. Such a species is $G = 1 + E_2 + E_4 + E_6 + \dots$. Applying (2.28), we can calculate the cycle index of D_E, and its isomorphism-types generating function:

$$
\begin{aligned}
\widetilde{D_E}(x) = {}& x + 3x^2 + 16x^3 + 218x^4 + 9608x^5 + 1540944x^6 + 882033440x^7 \\
& + 1793359192848x^8 + 13027956824399552x^9 + \dots
\end{aligned}
$$

3.3.2 Digraphs in which every vertex has outdegree k

The species of such digraphs is D_{E_k}. So we must solve $G + G' = E_k$ for G. The virtual species $E_k - E_{k-1} + E_{k-2} + \dots$ is such a solution. We summarize our results in Table 3.1.

3.3.3 Digraphs with outdegrees from a prescribed set

Given a specified set S of positive integers, we can enumerate digraphs in which all outdegrees are members of S. Consider the case $S = \{1, 3, 4\}$, for example. The species of digraphs in which all outdegrees are members of S is D_G, where $G = E_4 + E_3 + E_1$. A solution to $G_1 + G'_1 = E_4 + E_3 + E_1$ is $G_1 = E_4 + E_1 - 1$, and we obtain the following isomorphism-types generating function:

$$
\widetilde{D_G} = x^2 + 2x^3 + 19x^4 + 616x^5 + 93815x^6 + 39097411x^7 + 30749550146x^8 \dots
$$

			Outdegree		
n	1	2	3	4	5
2	1	0	0	0	0
3	2	1	0	0	0
4	6	6	1	0	0
5	13	79	13	1	0
6	40	1499	1499	40	1
7	100	35317	257290	35317	100
8	291	967255	56150820	56150820	967255
9	797	29949217	14971125930	111359017198	14971125930

Table 3.1: Digraphs of outdegree k on n vertices

Chapter 4

Graphs

4.1 G-graphs

In analogy with Section 3.1, we define a G-graph on a set U to be a graph with vertex set U, together with a G-structure on the set of vertices incident at each vertex. We denote by $\mathcal{G}_G^\circ$ the species of G-graphs in which loops and multiple edges are allowed, and by $\mathcal{G}_G$ the species of G-graphs in which loops are not allowed (but multiple edges are).

We note that since multiple edges are allowed, $\mathcal{G}_G$ and $\mathcal{G}_G^\circ$ are defined if and only if G is strictly finite.

To construct the species of G-graphs combinatorially, consider first the species $X \cdot G(Y)$ and $E(X \cdot G(Y))$. A structure of this latter species is pictured in Figure 4.1 with $G = E_3$. Creating a G-graph from such a structure amounts to pairing up the Y points to form edges, as shown in Figure 4.2. (We note that both loops and multiple edges are permitted by this construction.)

In species-theoretic terms this means that a G-graph is specified by both an $E(X \cdot G(Y))$-structure and an $E(E_2(Y))$-structure on a given set of points. The species of such pairs of structures is $E(X \cdot G(Y)) \times_Y E(E_2(Y))$. Since each edge consists of *two* Y-points, the number of Y points is twice the number of edges in the corresponding graph. In any case, setting $Y = 1$ gives the species $\mathcal{G}_G^\circ$:

$$\begin{aligned}
\mathcal{G}_G^\circ(X) &= \left. \left(E(X \cdot G(Y)) \times_Y E(E_2(Y)) \right) \right|_{Y=1} \\
&= \left\langle E(X \cdot G(Y)), E(E_2(Y)) \right\rangle_Y
\end{aligned} \tag{4.1}$$

Recalling Lemma 2.26, and observing that the operations involved are polynomial, we note that the map $G \mapsto \mathcal{G}_G^\circ$ is polynomial.

4.2 Removing loops

Removing loops from G-graphs is analogous to removing them from G-digraphs. For $n > 0$, let $G^{(n)}$ denote the result of applying the derivative operator to G n times. We have:
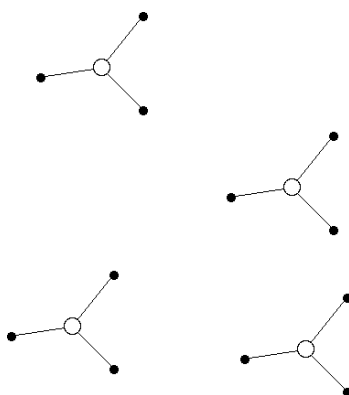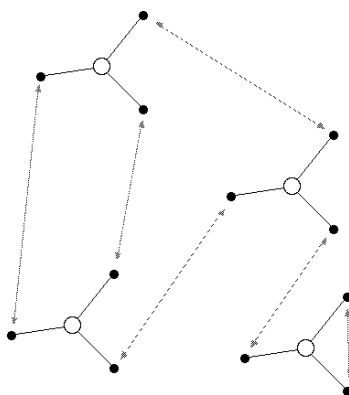
Figure 4.1: An $E(X \cdot E_3(Y))$-structure

Figure 4.2: A 3-regular graph

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Lemma 4.1. *If G is a species, then $\mathcal{G}_{G+G''+G^{(4)}+G^{(6)}+...} = \mathcal{G}_G^\circ$.*

Proof. A G-structure on the edges incident upon a vertex with a loop can also be thought of as a G''-structure on these edges with the loop removed. If two loops are present, we can regard a G-structure on the incident edges as a $G^{(4)}$-structure on these edges with both loops removed, and so forth. Thus, given a $\mathcal{G}_G^\circ$-structure on a set U, we can remove all the loops and replace the G-structures at those vertices with the same structures considered as G''-structures on two fewer objects, or $G^{(4)}$-structures on four fewer objects, etc. This gives a natural bijection from $\mathcal{G}_G^\circ[U]$ to $\mathcal{G}_{G+G''+G^{(4)}+G^{(6)}+...}[U]$. $\qquad\square$

Lemma 4.2. *The map $G \mapsto \mathcal{G}_G$ is polynomial.*

Proof. The proof is analogous to that of Lemma 3.3. Let M be a molecular subspecies of $\mathcal{G}_G$, where $G = \sum_{N \in \mathfrak{M}_1} a_N N$. Suppose M is homogeneous of degree m. An element of $M[m]$ consists of a graph $\mathcal{R}$ on $[m]$, together with a G-structure on the edges incident upon every vertex. Any G-structure is an N-structure, for some $N \in \mathfrak{M}_1$, colored in one of a_N colors.

Let $N_1, \ldots N_s$, be the elements of $\mathfrak{M}_1$ whose structures appear at the vertices of $\mathcal{R}$. Furthermore, let k_i be the number of distinct colors of N_i-structures which appear, and let $a_i = a_{N_i}$.

Let $T = \sum_{i=1}^s k_i N_i$. Then $\mathcal{G}_T$ is naturally isomorphic to a subspecies of $\mathcal{G}_G$, and any choice of k_i colors out of a_i yields a subspecies of $\mathcal{G}_G$ which is isomorphic to $\mathcal{G}_T$. Let p be the coefficient of M in the molecular series of $\mathcal{G}_T$. Then we have the following contribution to the coefficient of M in the molecular series of $\mathcal{G}_G$:

$$p \prod_{i=1}^s \binom{a_i}{k_i} \tag{4.2}$$

We will now show that the coefficient of M in $\mathcal{G}_G$ is in fact a sum of such terms.

Suppose M^* is another molecular subspecies of $\mathcal{G}_G$ which is isomorphic to M. Then M^* must be homogeneous of degree m. We can define s^*, $N_1^*, \ldots, N_{s^*}^*$, a_i^*, and k_i^* just as we defined their analogues for M, and let $T^* = \sum_{i=1}^{s^*} k_i^* N_i^*$. If $T^* = T$ then M^* is counted in (4.2). If not, let p^* be the coefficient of M^* in T^*; we obtain a new expression of the same form as (4.2), with all quantities replaced by their starred analogues.

To see that there are only finitely many possibilities for the species T^*, we use the fact that G is strictly finite (recall that $\mathcal{G}_G$ is only defined if G is strictly finite), and therefore has only finitely many molecular subspecies. $\qquad\square$

The map $G \mapsto \mathcal{G}_G^\circ$ is polynomial. Thus, Lemma 4.2 allows us to conclude that Lemma 4.1 holds for virtual species. To calculate the cycle index of $\mathcal{G}_G$, we can solve the equation $G_1 + G_1'' + G_1^{(4)} + \cdots = G$ for G_1, and calculate the cycle index of $\mathcal{G}_{G_1}^\circ$. We note that $G - G''$ is a solution.

4.3 Application: regular graphs

Consider the problem of enumerating k-regular graphs, that is, graphs in which every vertex has degree k. In our framework, these graphs are given by $\mathcal{G}_{E_k}$ or $\mathcal{G}^{\circ}_{E_k}$, depending on whether or not loops are allowed.

Consider $\mathcal{G}_{E_3}$, for example. Applying the methods of Section 4.2, we solve $G_1 + G_1'' + G_1^{(4)} + \cdots = E_3$ for G_1, obtaining $G_1 = E_3 - E_1$. We then use the combinatorial construction (4.1) to calculate the cycle index of $\mathcal{G}^{\circ}_{E_3 - E_1}$, which, by Lemma 4.1, is also the cycle index of $\mathcal{G}_{E_3}$. We obtain the following isomorphism-types generating function:

$$\widetilde{\mathcal{G}_{E_3}}(x) = x^2 + 3x^4 + 9x^6 + 32x^8 + 135x^{10} + \ldots \tag{4.3}$$

We note that the problem of enumerating 3-regular graphs arose in chemistry in 1966 (though the interest there was in connected graphs). Read [12] had solved the problem of enumerating 3- and 4-regular graphs in which loops, multiple edges, or both, are excluded, but only in the case in which either the edges or the vertices were unlabeled. The general problem was considered unsolved.

In fact, the Dutch mathematician Jan de Vries had enumerated k-regular graphs in the unlabeled case in 1891 (though not in graph-theoretic language), but his work did not become widely known for nearly a century. See [4] for the full story.

Asymptotic results are also known (see [1]).

4.4 Bicolored graphs

We recall that a *bicolored graph* is one whose vertices have been partitioned into two (non-empty) sets, such that vertices in the same set are never adjacent. We define a *bicolored G-graph* to be a G-graph in which the vertices have been so partitioned. To count bicolored G-graphs, we use the inner plethysm in Y.

Let $E^* = E_1 + E_2 + E_3 + \ldots$ denote the species of non-empty sets, and consider the species $E^*(X \cdot G(Y))$. A structure of $E_2 \boxtimes_Y E^*(X \cdot G(Y))$ consists of two such structures which share the same Y points (see Figure 4.3 for an example when $G = E$). Thus, $E_2 \boxtimes_Y E^*(X \cdot G(Y)) = \mathcal{B}_G(X, Y)$, where $\mathcal{B}_G(X, Y)$ is the species of bicolored G-graphs with vertices of sort X and edges of sort Y.

The techniques of Section 2.4.2 allow us to calculate the cycle index series of $E_2 \boxtimes_Y E^*(X \cdot G(Y))$, and thus its isomorphism-types generating function. For example, with $G = E$, we find:

$$\begin{aligned}
\widetilde{\mathcal{B}_E}(x, y) = {} & x^2(1 + y + y^2 + y^3 + y^4 + y^5 + \ldots) \\
& + x^3(1 + y + 2y^2 + 2y^3 + 3y^4 + 3y^5 + \ldots) \\
& + x^4(2 + 2y + 5y^2 + 7y^3 + 12y^4 + 15y^5 + \ldots) \\
& + x^5(2 + 2y + 6y^2 + 10y^3 + 21y^4 + 32y^5 + \ldots) \\
& + \ldots
\end{aligned}$$

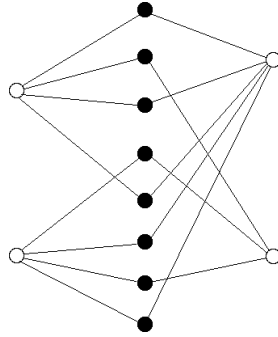Figure 4.3: An $E_2 \boxtimes_Y E^*(X \cdot E(Y))$-structure

We note that the problem of enumerating bicolored graphs, in which multiple edges are not allowed, was solved by Parthasarathy in [11].

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